ONTARIO
TEACHERS' MANUALS

ARITHMETIC

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ARITHMETIC

PUBLIC AND SEPARATE SCHOOL COURSE OF STUDY

DETAILS

Purpose: To enable the pupil to acquire facility, accuracy, and promptness in the use of numbers in calculation, and to learn to reason correctly.

FORM I. JUNIOR GRADE

The Objective Development of the Number Idea (Chap. II)

The Numbers 1 to 10 Inclusive (Chap. III):

1. Objects counted and measured to the limit of the above numbers (Chap. III).
2. Objects grouped in twos, threes, etc., and their measure expressed in terms of the numbers taught. Thus the measure of seven may be expressed as 2 threes and 1, 3 twos and 1, 1 four and 3, etc., (Chap. III).
3. Objects grouped in tens. Thus, 3 tens and 2, 4 tens and 7, 1 ten and 5, etc., (Chap. III).

Numeration and Notation (Chaps. III, IV):

1. Counting to 100 (Chaps. III, IV).
2. Numeration and notation of units and tens taught objectively with special emphasis on their position and on their consequent relation to one another (Chaps. III, IV).
3. Thorough drill in numeration and notation to 100.

Combinations:

1. Analysis and synthesis of numbers to 10 taught objectively (Chap. III).
2. Thorough drill on the following types:
    (*a*) 5+2=?
    (*b*) 5+?=7
    (*c*) ?+?=7

Additions:

1. Single-column addition, the sum not to exceed 10 (Chap. III).
2. Addition of tens (1 ten, 2 tens, 3 tens, etc.) to any number, the sum not to exceed 100 (Chap. IV).
3. Addition between successive tens. Thus, 12+5+2; 21+3+4; etc., (Chap. V).
4. Addition of addends grouped in tens. Thus, 2+3+5+8+2+6+3+1; 24+5+1+3+5+2+6; etc., (Chap. V).

Subtraction (oral):

1. Subtraction of any single-digit number from any number less than 100 of which the unit's digit is greater than the number subtracted (Chap. III).
2. Subtraction of any ten (1 ten, 2 tens, 3 tens, etc.) from any number less than 100 (Chap. III).

Easy Oral Problems:

These to involve one step and to be related to the lives of the pupils.

FORM I. SENIOR GRADE

Course of Junior Grade Reviewed and Extended

Numeration and Notation:

1. Numeration and notation of units, tens, hundreds, and thousands taught objectively, and so as to show the place value of the digits (Chap. VI).
2. Numeration and notation to six figures (Chap. VI).
3. Position of numbers in the number series or the arrangement of numbers in order according to their value, from least to greatest, etc., (Chap. VI).

Combinations:

1. Analysis and synthesis of numbers to 20 (Chaps. III, IV, V).
2. Drill on types:
    (*a*) 5+9=?
    (*b*) 5+?=14
    (*c*) ?+?=14
3. Application of these combinations to find the sum of any two numbers one of which is less than 10 and the other less than 100 (Chap. V).
4. Thorough drill on combinations taught, special attention being given to such combinations as: 3 and 6, 13 and 6, 43 and 6, 5 and 7, 35 and 7, etc., (Chap. V).

Additions:

1. Thorough drill in combinations and in single-column additions in order that addition may become automatic (Chap. V).
2. Two- and three-column additions (Chap. V).
3. Additions by 1's, 2's, 3's, 4's, etc., to sums not exceeding 100; thus laying basis for multiplication tables (Chap. V).

Subtraction:

1. Subtraction facts to 20 taught in connection with addition facts (Chaps. III, V, VII).
2. Oral exercises in the subtraction of any number less than 10, from any number less than 100 (Chaps. III, V, VII).
3. Subtraction of any numbers not exceeding three digits (Chap. VII).

Concrete Units:

Objective treatment of one cent, five cents, twenty-five cents, fifty cents, dollar; pint, quart, gallon, peck; inch, foot, and yard (Chap. VIII).

Easy Oral Problems:

These to involve not more than one step in their solution and to be related to the lives of the pupils.

FORM II

Review and Extension of Work of Form I

Numeration and Notation to Nine Figures (Chap. V)

THOROUGH DRILL ON ADDITION AND SUBTRACTION:

Use, for the most part, numbers represented by not more than five digits; give special attention to securing accuracy, facility, and rapidity (Chaps. V, VII).

MULTIPLICATION (Chap. IX):

1. Multiplication table.
2. Multiplication by numbers, from 1 to 10 inclusive.
3. Multiplication to find continued product where the multipliers are single-digit numbers.
4. Multiplication by 10, 20, 30, etc.
5. Multiplication by any two-digit number.
6. Multiplication by 100, 200, 300, etc.
7. Multiplication by any three-digit number.

DIVISION (Chap. X):

1. Division by numbers 1 to 9 inclusive.
2. Division by 10, 20, 30, etc.
3. Division by any two-digit number.
4. Easy factoring (Chap. XI).

CONCRETE UNITS (Chap. VIII):

1. Objective development of pint, quart, gallon, peck, bushel; ounce, pound. Relation of minute, hour, week, month, year.
2. Use of these and previous units in oral and written problems not involving more than two operations in their solution.

ROMAN NOTATION TO 1000 (Chap. IX)

PROBLEMS:

1. Practice in solving, orally and in writing, easy problems related to the lives of the pupils and which involve not more than two operations.
2. Drill in reading problems and in stating solutions orally.
3. Practice in solving easy problems involving the applications of all topics taught, emphasis being placed on clear, logical statements in the solution of problems, and on accuracy and rapidity in mechanical work.
4. Daily practice in Oral Arithmetic.

FORM III. JUNIOR GRADE

REVIEW:

The work of previous Forms; special attention to be given to neatness, accuracy, and rapidity in mechanical work, and to clear, logical statements in problems.

NOTATION AND NUMERATION REVIEWED AND COMPLETED

ADDITION AND SUBTRACTION COMPLETED; THOROUGH DRILL AND MUCH PRACTICE

MULTIPLICATION AND DIVISION:

1. Full and thorough drill to secure accuracy, facility, and rapidity.
2. Simple bills and accounts.
3. Aggregates and averages.
4. Continued products.
5. Finding by trial an exact divisor of a number.
6. Division by two factors, where no remainders occur (Chap. XI).

TABLES (Chap. XIII):

1. Tables of money, time, length, capacity, weight, area, and volume.
2. Application of these tables to reduction, descending and ascending.

FRACTIONS (Chap. XV):

Inductive treatment of vulgar fractions in common use:

1. Meaning.
2. Notation and numeration.
3. Method of finding fractional parts of any quantity or number.
4. Writing tables of equivalents.
5. Addition, subtraction, and comparison.
6. Expression of improper fraction as mixed number, and converse.

PROBLEMS:

1. Practice in solving problems applying the topics taught.
2. Practice in framing problems from data furnished by pupils.
3. Daily drill in Oral Arithmetic.
4. Emphasis should be placed upon clear, logical statements in solutions, upon neatness, and upon accuracy and rapidity.

FORM III. SENIOR GRADE

REVIEW:

The work of the Junior Grade; special attention to be given to neatness, accuracy, and rapidity.

Compound Rules (Chap. XIII):

1. Tables completed.
2. Reduction completed.
3. Compound addition, subtraction, multiplication, and division.

Factors and Measures (Chaps. XI, XVI):

1. Method of finding prime factors and of using these to find the common factors and the highest common factors of two numbers.
2. Division by factors and method of finding true remainder.
3. Cancellation.

Multiples (Chap. XVI):

1. Easy method of finding multiples and common multiples of small numbers.
2. Thorough treatment of least common multiple.

Fractions and Mixed Numbers (Chap. XV):

1. Review work of Junior Grade.
2. Multiplication by an integer.
3. Simplification of compound fractions involving but two fractional parts.
4. Division by an integer.
5. Multiplication by a fraction.
6. Division by a fraction.

Simple Percentages (Chap. XVIII):

Their relation to fractions.

Personal and Cash Accounts

Simple Measurements (Chap. XIV):

1. Perimeters and areas of rectangular figures.
2. Volumes of rectangular solids.
3. Board measure.

Problems:

Oral and written problems as for the Junior Grade.

FORM IV. JUNIOR GRADE

Review of Work of Form III

Measures and Multiples:

1. Treatment of greatest common measure completed.
2. Treatment of least common multiple completed.

Vulgar Fractions:

1. Operations completed.
2. Complex fractions.
3. Special drill on changing a fraction to an equivalent one the denominator of which is given.
4. Special drill in expressing one quantity or number as a fraction of another.

Terminating Decimal Fractions (Chap. XVII):

1. Relation to vulgar fractions.
2. Notation and numeration; relation to that of integers.
3. Operations of addition, subtraction, and multiplication.

Simple Percentage (Chap. XVIII):

1. Relation to fractions.
2. Method of converting [illegible] into a fraction and converse.
3. Application to simple problems of the three types.

SIMPLE CASH ACCOUNTS SHOWING REVENUE AND EXPENDITURE

MEASUREMENTS (Chaps. XIV, XX):

1. Perimeter and area of rectangular figures.
2. Volume of rectangular solids.
3. Solution of practical problems involving the measurement of land, walls, floors, roofs, pavements, gradings, excavations, and such like.
4. Solution of problems framed from data obtained from actual measurements made by the pupils.

PROBLEMS:

Solution of such oral and written problems as find direct application in ordinary life; special attention to be given to neatness, accuracy, and logical arrangement of steps.

FORM IV. SENIOR GRADE

REVIEW:

The work of the Junior Grade; special attention to be given to neatness, accuracy, and rapidity in mechanical work and to clear, logical arrangement of steps in the solution of problems; emphasis to be placed upon Oral Arithmetic.

TERMINATING DECIMALS (Chap. XVII):

1. Operations completed.
2. Reduction of decimals to vulgar fractions, and vice versa.
3. Problems to illustrate use and application of decimals.

PERCENTAGE (Chap. XVIII):

1. Review of work of Junior Grade.
2. Application of percentage to:
    - (*a*) Commission
    - (*b*) Trade Discount
    - (*c*) Gain and Loss
    - (*d*) Insurance
    - (*e*) Taxes
    - (*f*) Simple Interest
    - (*g*) Bank Discount

MEASUREMENTS (Chap. XX):

1. Perimeters and areas of rectangular figures and of right-angled triangles.
2. Volumes and areas of surfaces of rectangular solids.
3. Problems involving these topics.

ACCOUNTS:

1. Cash accounts showing revenues and expenditures.
2. Bank accounts showing deposits and withdrawals.

BUSINESS FORMS:

1. Bills
2. Accounts
3. Receipts
4. Cheques
5. Promissory Notes

PROBLEMS:

Solution of such problems oral and written as find direct application in ordinary life.

ARITHMETIC

CHAPTER I

INTRODUCTION

SCOPE AND EDUCATIONAL VALUE

Arithmetic includes the theory of numbers, the art of computation, and the application of numbers to science and business.

Next to language study, arithmetic has held the leading place in the curriculum of the Public Schools. Until quite recently its importance was based upon its prestige as a subject unsurpassed for mental discipline. It was regarded as "the logic of the Public School". It must, however, be accepted as an established fact in the science of education that no subject possesses a purely formal disciplinary value. Training in one subject of study can give mental aptitude for other subjects only in so far as that subject furnishes knowledge or functional ideas which enter as elements into the other subjects. The recognition of this principle has had a tendency, during the last few years, to place the emphasis upon the practical value of arithmetic, that is, upon its application to the ordinary affairs of life. It is this aspect of the subject which, in the early stages at least, is most likely to arouse interest and to determine progress. Nevertheless, a teacher cannot disregard the fact that the study of arithmetic furnishes opportunities for the development of most useful mental habits within a very wide field. It affords exercise in logical analysis, in the discovery of "associated relationships" between stated conditions, in the orderly arrangement of these relationships, in accuracy of computation, in concentration of attention and the exercise of memory upon a special kind of material, as well as in conciseness, in clearness, and in accuracy of expression.

These habits, however, are the result not of the material used by the pupil but of the manner in which that material is used by both the teacher and pupil. They are incidental products due to proper guidance and direction, that is, to the skill of the teacher and the insistence upon the employment of rational methods.

CORRELATION WITH OTHER SUBJECTS OF STUDY

Owing to the extensive use of special symbols, arithmetic is likely to be considered as having little relation to other school subjects, but reflection will show that its inter-relationships are very extensive. All other subjects of study have a quantitative aspect. This aspect is not so marked in literature, in history, and in the language studies; but all advances in the other mathematical studies and in the natural and the physical sciences, both in their development and in their application to industries, are directly dependent upon quantitative determinations and arithmetical calculations. In the Elementary Schools, its most natural relations are with the practical exercises in mathematical and physical geography, and with the activities of nature study, school gardening, agriculture, manual training, household science, and book-keeping. Illustrative examples of these relations will be found throughout this Manual.

Arithmetic also, on the expression side, is connected with composition. Its operations demand concise, clear, and accurate expression.

TEACHING STAGES

The teaching of arithmetic may be divided into two stages—the Rational and the Empirical—the latter preparing the way for the former. The rational presents the logic of the subject and represents the *problem* stage, while the latter represents the *mechanical* stage and provides the pupil with a "rich collection of concrete experience" from which he derives a working notion of number, of its notation and numeration, and of the elementary operations of addition, subtraction, multiplication, and division.

GENERAL METHOD

The method of teaching any subject must accord with certain principles determined by the nature of the learning process. These principles of general method are discussed in the *Science of Education* They may be briefly summed up in the following statement:

1. *The Problem.*—The acquisition of new knowledge is initiated by some problem in the solution of which the learner takes an interest.

2. *A Selecting Process.*—The problem is analysed, and ideas are selected which are felt to possess value relative to the solving of the problem.

3. *A Relating Process.*—These relevant ideas are combined into a plan of solution.

4. *Expression.*—The new knowledge is expressed and applied.

These steps, which are involved in the acquisition of knowledge in any field, are clearly recognized in the typical arithmetic lesson.

It is probably unnecessary to state that the word "problem" in this summary is used in a general sense and, when applied to arithmetic, has reference to questions relating to the development of principles and rules, as well as to those in the application of these principles in the working of the exercises.

The Manual on the *Science of Education* gives full illustrations of the application of these steps in arithmetic recitations. Further applications are found in the illustrative lessons in this Manual.

As a rule, an arithmetical recitation covering a complete lesson unit follows the inductive-deductive type. Principles and rules are developed inductively and applied deductively.

At each step oral arithmetic should precede written arithmetic, and throughout the Course, accuracy and facility in computation are essential.

The model lessons and the illustrations of principles presented in this Manual are to be considered as suggestive and not directive. The intelligent teacher will adopt the method by which he feels he can do the most effective work. Where a choice of methods is offered, it is advised that but one should be taught to young pupils, and that the method in use in a class should not be changed, unless clearly necessary.

CHAPTER II

NUMBER: OBJECTIVE BASIS

DEVELOPMENT OF THE IDEA OF NUMBER

THE content topic of arithmetic is number and its applications. A child sees apples on a plate, and recognizes a Baldwin, a Russet, a Pippin, and a Spy. He has observed individual qualities that enable him to distinguish one apple from another. This is the *qualitative* view. Dropping out of sight individual qualities in the apples, he counts them and says there are four apples. This is the *quantitative* view. In counting, only the quantitative view is considered, and the result is stated numerically. Number is the measure of quantity.

At first, a child perceives all differences of number as mere differences of magnitude—of greater and less. Later, he learns to distinguish between magnitude, which is continuous quantity, and number, which is discrete quantity. In number he discriminates "one" and "many" long before he can distinguish "two" from "three".

When he begins to count, he analyses a group of objects, which he has perceived as a plurality, into its separate parts, or units. In order to distinguish number from form, the objects in this group are rearranged in various ways and counted. Other groups, similar only in respect of number, are counted, and gradually, through comparison and abstraction, the number idea is developed, number relations are established, and the use and meanings of the number symbols made clear.

In general terms, number arises out of the necessity for evaluating quantity in terms of some more easily comprehended quantity. The different stages in its development may be stated somewhat as follows:

1. The recognition of quantity;
2. The comparison of quantities;
3. The division of quantity into parts;
4. The equalization of these parts;
5. The combination of these parts to form the whole;
6. The expression of that whole in terms of its relation to its equal parts.

Or, more briefly, there is first the notion of a *quantity*, then of a *unit*, then of a *number*.

NUMBER THROUGH OBJECTS

To the child the idea of number undoubtedly comes in a concrete way, and the natural and normal starting-point for its development must be found within the child's own experience, which must be made as broad and varied as possible. His introductory lessons will, therefore, be connected with familiar objects which he can handle, count, measure, and compare, with his toys and playmates, and with the multitude of other objects which constitute his environment.

The fundamental operations in arithmetic should be performed with simple objects and illustrated with other objects till the idea or principle becomes clear. Then the operations should be performed symbolically with numbers. The objects by means of which principles are illustrated should have few accessories. It is the quantitative aspect that the pupil is to observe; the qualitative aspect, if prominent, distracts his attention.

MATERIAL FOR OBJECTIVE WORK

Under our Regulations each school should have:

1. A numeral frame and an adequate supply of material for teaching numbers;
2. A set of mensuration surface forms and geometrical solids;
3. A black-board set for each class-room, a protractor, a triangle, a pair of compasses, two pointers, a graduated straight edge;
4. A pair of scales with weights to weigh from half an ounce to at least four pounds;
5. A set for measure of capacity—pint, quart, gallon;
6. A set for linear measure—inch, foot, yard, tape-line;
7. A set for square and cubic measures.

Abundance of suitable material for the first lessons in the subject, such as splints, kindergarten sticks of different lengths and colours, shoe pegs, toothpicks, beads, horse-chestnuts, large beans, large smooth buttons, smooth pebbles, checkers, dominoes, wooden discs, cubical blocks, kindergarten tablets, etc., can be procured at very little cost; while the teacher with any degree of resource can always find a piece of wood from which with a handsaw and knife he can make for himself very satisfactory sticks or blocks, uniform in size and shape—circular, rectangular, or triangular. Besides all these the teacher always has at hand such interesting objects as the children themselves, school desks, the rows of desks; pictures in books, on the wall or on the black-board, and the number pictures.

Blank forms for accounts, receipts, deposits, cheques, notes, drafts, etc., for use in lessons in commercial arithmetic, can always be obtained from business houses.

USE OF CONCRETE MATERIAL

Directions for the use of concrete material are given at length in the discussion of various topics throughout this Manual. In general, it may be said that objects are used:

1. To enable the pupil to make discoveries and test results;
2. To enable the teacher to make clear to the child, whose vocabulary is limited, what he is attempting to teach;
3. To enable the pupil to show the teacher that what has been taught is clearly understood.

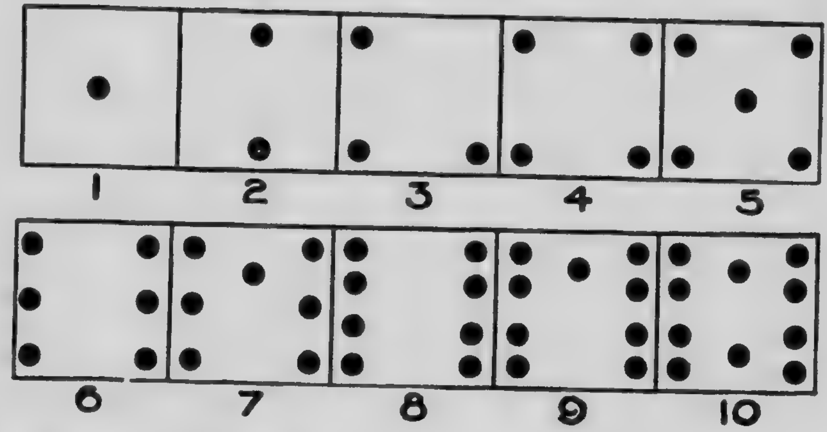
At first, objective material should be used very freely and the work should be almost wholly concrete; gradually the symbolic phase of the subject should be introduced and the lessons made more abstract in character. But even in the more advanced departments, in both mensuration and commercial arithmetic, it is advisable to give the subject, where possible, a concrete basis.

USE OF NUMBER PICTURES

A beginner usually finds difficulty in passing from the concrete to the symbolic phase of the subject. The teacher will find the use of number pictures of assistance to him in leading the pupil to understand the use and to appreciate the value of the symbols.

These pictures are dots or marks arranged in such a systematic way that their number is readily grasped. Dominoes answer well as number pictures.

The dots of the number pictures may be used at first as objects for counting, etc.; when thus used they serve to show that objects can be more easily counted in some arrangements than in others. Later they may be used as



symbols to stand for other objects. For example, the number picture four may be used to stand for four splints, four nuts, or for four cubes. When the pupils are required to represent in this way various numbers of things by the corresponding number pictures, they are led gradually to feel the necessity for symbols. The written symbols (figures) may then be introduced. The number pictures have the advantage, in the beginning, that the numbers which they symbolize are represented in the pictures themselves; with the written symbols this is not the case. For example, the number picture five has some "fiveness" about it, while the figure 5 has none.

The written symbols must not be introduced too early. This caution is necessary, because with many teachers the tendency is to get away from the concrete too quickly and to make the lessons in arithmetic exercises in the manipulation of figures. This tendency is specially strong when the teacher is not skilled in assigning a variety of seat work. It is easy to give "sums", and the pupils usually get a surfeit of them. On the other hand, it must constantly be kept in mind that objects are used to enable the pupil eventually to do without them.

CHAPTER III

NUMBERS FROM 1 TO 10

FIRST STEP

TO DISCOVER PUPIL'S NUMBER ATTAINMENT

THE first lessons in any subject should be planned with the object of making the pupil feel at ease in the schoolroom and of getting him to express himself freely. In arithmetic these first lessons should also enable the teacher to ascertain the pupil's number attainments and discover his quantity vocabulary. If number is the measure of quantity, then the pupil must have some idea of the latter before much progress can be made in the former. What that idea is, and how clear it is, will be made manifest in the pupil's language. He will express it first vaguely in such general terms as "enough", "lots", "plenty", "a big pile", "a small bit", "a long way", "a short piece", "far away", "many", "few"; then more definitely by comparison—"this is bigger, heavier, higher, longer, smaller, more, less, than that"; and finally, by telling the exact measure, as "five", "three", "seven", etc.

It is evident, therefore, that the first step in teaching number will consist of a series of skilfully conducted conversations between pupil and teacher, by which the teacher will incidentally be put in possession of the pupil's knowledge of quantity and number. These conversations will be based on the pupil's activities, his games and amusements, his family, his playmates, and his pets.

The pupil having shown that he is able to make comparisons between quantities, definite exercises therein may be given him with objects—sticks, blocks, paper strips, etc., of different sizes.

Illustration

"Take a pile of these blocks. Hold up the smallest of them; the largest of them; one which is smaller than this one and larger than that.

"Take some of the blocks away. Which make the bigger pile, the blocks you took away or those that were left? Why?

"Put your blocks in a row. Pick out the smallest and put it at the left side of the desk. Now put the next smallest beside it, the next smallest, etc.

"Put your blocks again in a pile. Now put other blocks with them. Place the blocks again in line from the smallest (or least) to the largest (or greatest).

"Place these sticks in the same way. Place the pupils in the class in the same way.

"Pick out a strip of paper. Pick up a strip which has more paper in it than there is in this one; another strip which has less in it.

"How much more paper has this strip than that? Make two strips the same size, that is, make them equal."

By some such exercises as these, the pupil is made familiar with many of the arithmetical terms which he must early understand—greater, smaller, least, larger, more, less, equal.

The teacher will now endeavour to ascertain how far the pupil can count.

Illustration

"What pets have you at home? What is your dog's name? What is your pony's name? Have you only one horse? How many have you? What are their names? How many cows have you? What are their names? How many hens do you keep? How many eggs do you get each day?

"You count well. How many boys are there in this class? How many girls? How many children? How many children in the room? Count the number of splints in this bundle, etc."

The foregoing is work for kindergarten pupils and may not require much time in the Elementary School classes, but it gives the teacher valuable information. It will be found that most children before entering school can determine the number of objects in a group up to four, and say the number series at least as far as ten, and not a few will be able to count to one hundred or more. It will be necessary, however, to make sure that in counting the pupil knows the number as a whole group and not as one of a group, that is, he must think of six as six ones and not as the sixth one.

SECOND STEP

TO EXTEND PUPIL'S KNOWLEDGE OF NUMBER SERIES

Beginning then with what the pupil can already do, the next step will be to extend his knowledge of the number series. Let it be assumed that he knows four. He will then be asked to apply his knowledge in various exercises.

Illustration

"Take up four splints. Put four books on the table. You have now as many books as splints.

"Pick up as many pegs as you have splints. Take four steps; now take another four steps. Put four strokes on the board.

"Put these blocks in bundles with four in each bundle. How many bundles have you? How many fours have you?

"Make four piles with three in each. How many threes have you? Make three piles with four in each. How many fours have you?

"Place out one bead, now two beads, next three, and then four.

"How many more is two than one, three than two, four than three? Count these pegs by threes. (Two threes and one more, three threes and two more, etc.) Count this bundle of pegs in fours."

TEACHING THE NEW NUMBER, FIVE

The teacher now gives the pupil five splints, five tablets, five buttons, five beads, five pebbles, five checkers, etc., and proceeds to teach the number five as follows: "Count these by twos. (Two twos and one more) Count these by threes. (One three and two more) Count them by fours. (One four and one more)

"What number comes after one? How many more is two than one?

"What number comes after two? How many more is three than two? What is the number after three? How many more is four than three?

"Now the bundle of splints had in it one more than four and the number of splints will be the next number after four. It is called five.

"Now pick out five splints, blocks, beads, pegs, books, boys, etc. How many have you?

"Take five steps and count while stepping. Tap on the board five times, that is, give five taps.

"Put these beads in piles with five beads in each. Make four piles with five in each pile. Make five piles with three in each. How many piles are there? How many threes are there?

"Now count these pegs by fives. (Three fives and two more. Five fives and four more, etc.)"

The succeeding numbers up to ten are taught similarly.

NAMING THE NUMBERS TO ONE HUNDRED

Practice in measuring, grouping, and counting objects by tens, follows. The pupil expresses his results thus:

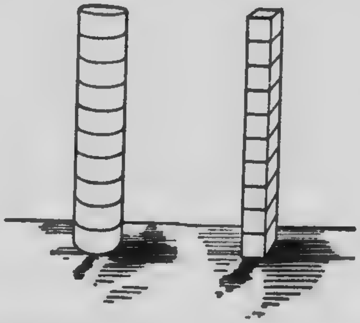
Six tens, three tens, five tens, one ten, eight tens, etc.; four tens and five, seven tens and one, nine tens and three, one ten and six, one ten and one.

He is then given the names of the tens—twenty, thirty, forty, etc., and given practice in using them as in—twenty and three, sixty and one, seventy and five, but still uses "one ten and four".

He then drops the "and" and, finally, is given the name of the numbers from ten to twenty, and so completes his counting from one to one hundred, beyond which he will experience little difficulty.

Concurrent practice should be given in counting various things in the school-room—the number in the class, the number of window-panes, the number of rows of seats, and in counting, on the numeral frame or with cubes, and such like, to the limit of his ability.

If the teacher has checkers, cylindrical blocks two inches in diameter and one inch high, or one-inch cubes, the pupil may build towers and count the height of each. He may then tear them down in parts and count the height of what remains.



It is not intended that the number series as far as 100 should be completely taught before any other topic in arithmetic is approached. When the number series to ten is reached, the addition and subtraction facts of these numbers, as well as their symbols, may be taught, and counting may be advanced as necessity requires, or opportunity permits.

THIRD STEP

MEASURING AND BUSY WORK

When the pupil can count to ten, he may be taught to measure with cord, strips of paper, his hands, his feet, his pace, his ruler, and with kindergarten sticks of definite but different lengths. Through these exercises, the "times" notion is developed, and the pupil is led to use the term intelligently.

At this stage, the teacher finds his greatest difficulty to be the assigning of suitable seat work. During the first month, at least, no symbols should be used. The work is oral and objective, and the whole time is devoted to counting, grouping, and measuring. For busy work, therefore, it will be necessary to devise ways in which the pupil can handle for himself, in a systematic manner, the materials to be grouped, counted, or measured. For this purpose the following suggestions may prove helpful:

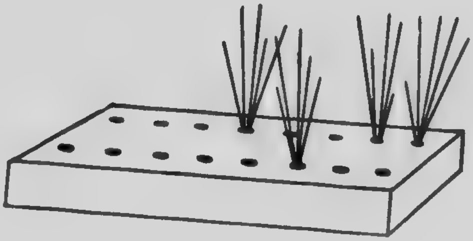


Fig. 1

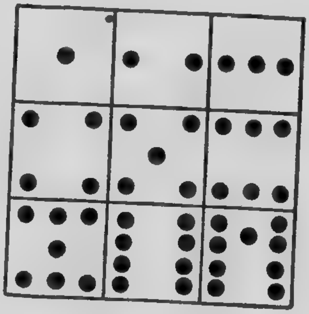


Fig. 2

1. Provide for each pupil a strip of wood 1 in. thick, 9 in. long and 4 in. wide, with two rows of 8 holes $\frac{3}{4}$ in. deep, and with this a small bundle of thin kindergarten splints, say, $2\frac{1}{2}$ in. long. These may be used as in Figure 1.

2. Divide a slate into rectangles and fill them in as in Figure 2.

3. Provide circular and square discs which may be placed together as in Figure 2.

4. Provide kindergarten sticks of different lengths, 1 in., 2 in.... 10 in.; place these and draw straight lines the same lengths, thus:

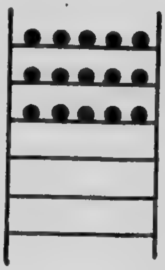


Fig. 3

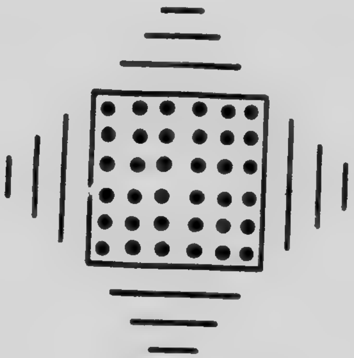


Fig. 4

5. Make on slates the accompanying diagram, Figure 3, with any named number of dots in the rows.

6. Draw, with pencils or crayons, dots and lines, as in Figure 4, with any named number of dots and lines.

Draw, with pencils or crayons, nests with one, two, [illegible] eggs.

8. Draw groups of birds, flowers, soldiers, crosses, animals, etc., with any named number in a group.

9. Make the number pictures referred to in Chapter II.

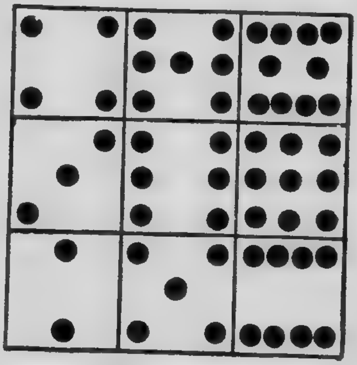


Fig. 5

The teacher should keep (*a*) a set of these on a large card 14 in. by 22 in., and (*b*) a single card 6 in. by 9 in. for each number, as in Figure 5.

FOURTH STEP

SYMBOLS AND ADDITION FACTS

The pupil can now count to 10 or beyond it. He has the perception of the first ten numbers, that is, he knows them as wholes; is able to separate 1, 2, 3, 4, ... or 10 objects from a group containing a greater number; can tell how many objects in a given group, if the number does not exceed ten; and can arrange any number of objects in uniform groups, each group containing ten or less. He can also represent these numbers by number pictures. He may now be given the symbol for each, and drilled in the rapid recognition of it; after which the addition and subtraction facts of these numbers should be taught.

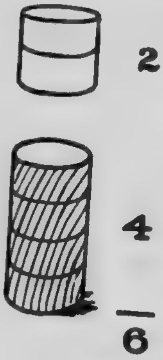
From the first, the pupil should be taught to write the *correct* forms of the symbols. These will be found in the authorized *Ontario Writing Course.*

ADDITION FACTS OF SIX

Illustration

Provide cylindrical blocks, cubes, beads, splints, etc. Review the addition facts of five, and drill on the perception of six.

"Pick out six splints. Tap the bell six times. Take six steps. Stretch the right hand six times. Make six bundles each of six cubes. Make patterns of six lines each. Pick out the six number-card."



Now set up, with cylindrical blocks, a cylinder six blocks high. Let the pupil separate it into any two cylinders. Different divisions will be made by different pupils. All are right. Select one combination, say 4 and 2, and have all the pupils divide their first cylinder like the one selected.

"How high was the first cylinder? How high is each of the two cylinders? Then, what two cylinders make the six-block cylinder? 4 blocks and 2 blocks are how many blocks?

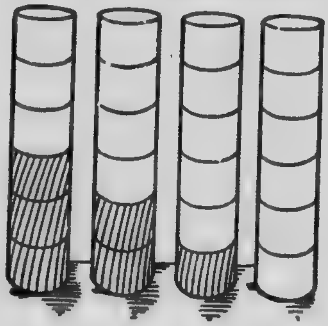
"Take six cubes and separate them in the same way as the blocks were separated. 4 cubes and 2 cubes are how many cubes? 4 splints and 2 splints? 4 and 2?"

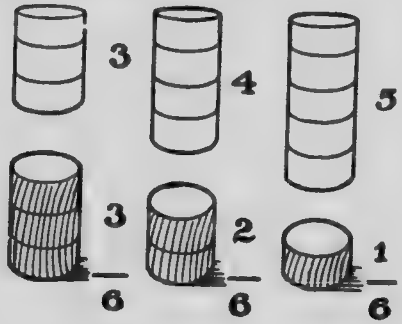
The groups into which some of the other pupils divided their cylinders at first, such as 3 and 3, are now taken, and dealt with in the same way, until all the combinations of 6 are found.

The pupil discovers each fact, states it orally, and is then told that it is written thus:

$$\begin{array}{r}4\\ \underline{2}\\ 6\end{array}\qquad\begin{array}{r}2\\ \underline{4}\\ 6\end{array}\qquad\begin{array}{r}3\\ \underline{3}\\ 6\end{array}\qquad\begin{array}{r}5\\ \underline{1}\\ 6\end{array}\qquad\begin{array}{r}1\\ \underline{5}\\ 6\end{array}$$

The diagrams below will explain the lesson:





Various groups of six objects, other than blocks, are now taken, such as books, children, pictures, etc. These are divided into two groups, and the addition facts of 6 are repeatedly verified.

These facts are to be committed to memory, and the pupil should be able to tell, without hesitation, what pairs of numbers added together make 6, and what number has to be put with a named number to make 6. In other words, he must be able to answer promptly each of the following questions:

1. 4+2=?
2. ?+2=6
3. ?+?=6

The various exercises with splints, blocks, number pictures, etc., will help the pupil to remember these addition facts, but for perfect memorization special attention must be given to, and special stress placed upon, drill with number symbols— first in such sums as $\begin{array}{r}3\\3\\\hline 6\end{array}$ $\begin{array}{r}4\\2\\\hline 6\end{array}$ $\begin{array}{r}5\\1\\\hline 6\end{array}$; then in the addition table, thus: 5+1=6, 4+2=6, 3+3=6; and lastly in rapid oral exercises with devices to be suggested later.

APPLICATION, ORAL AND WRITTEN

As each fact is learned, it may be applied in the oral solution of problems such as the following:

1. Mary had 5 cents and then earned 1 cent. How many cents had she then?

2. There are 3 birds in one tree and 3 in another. How many birds are there in both trees?

3. Three white kittens and three black kittens are how many kittens?

4. There are 2 pigs in the pen and 4 in the yard. How many pigs are there altogether?

5. Mary has 2 pink roses in her hand and 3 in her hat. How many roses has she?

6. Fred has 2 blue marbles and 2 red ones. How many marbles has he?

Addition sums may now be introduced for class work. They will be used for seat work as soon as the "facts" are so well known that there will be no danger of the pupil acquiring the method of "counting".

From the outset, the exercises must be carefully graded, and should be so arranged that the exercises on each new number will involve a review of those on previous numbers. The addition of two addends is the first exercise. After such facts as $\begin{array}{r}2\\ \underline{3}\\ 5\end{array}$ have been illustrated by the teacher, by means of number pictures or otherwise, the pupils learn them in the form of an addition sum with two addends.

Then will follow three addends, then four, and so on.

Illustration

1. $\begin{array}{ccccccc} 1 & 1 & 2 & 1 & 3 & 2 & 4 \\ \underline{1} & \underline{2} & \underline{1} & \underline{3} & \underline{1} & \underline{2} & \underline{1} \end{array}$ etc.

2. $\begin{array}{cccccccc} 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 3 & 2 & 1 \\ \underline{1} & \underline{2} & \underline{1} & \underline{1} & \underline{3} & \underline{1} & \underline{2} & \underline{2} \end{array}$ etc.

From the first, answers should be checked, and accuracy and rapidity kept as an important aim.

FIFTH STEP

REVIEW: SUBTRACTION FACTS TAUGHT

When the addition facts of the numbers from 1 to 10 have been learned, these facts should be reviewed, and the subtraction facts taught. This should be done by means of questions such as:

"What two parts can be made out of 6? What must be added to 4 to make 6?"

It can be shown objectively that taking 4 from 6 is in reality finding the number which added to 4 will make 6, and so subtraction may be considered as the operation of separating a number into two parts, one of which is given; the given part being taken away, the other remains. The connection between addition and subtraction being made clear, the memorization of the subtraction facts is rendered easy, as every addition fact has its corresponding fact in subtraction.

CHAPTER IV

NUMBERS FROM 10 TO 100

NUMERATION AND NOTATION

OBJECTS SELECTED

OBJECTIVE material should now be selected with a view to having the objects used in counting as nearly alike in size and shape as possible, in order that any ten objects will be actually 10 times one object. For the ones, the one-inch enlarged kindergarten stick or the one-inch cylindrical block will be found satisfactory; while, for the tens, bundles may at first be made of the ones, and these may, for convenience, be later replaced by the ten-inch kindergarten sticks, the pupils having by frequent measurement discovered that each of these sticks will make a bundle of ten of the small sticks.

OBJECTIVE SYMBOLS

By placing one of the small sticks on or beside each object in a set of objects—such as books, pens, pencils, marbles, desks, children—and then removing the objects, the pupils will see that they can tell, by counting the sticks instead of the objects themselves, how many objects there are, and that these sticks may, therefore, be used as "counters" to enumerate all kinds of objects. They will see that these sticks may take the place of the various kinds of objective material hitherto employed in our number work, and it will be understood that, in whatever way we arrange or combine the sticks, all other objects to be counted may be arranged or combined in a similar way.

DECIMAL SYSTEM

Counting to 100 has already been taught. Objects have been grouped in tens and the number expressed first in tens as "six tens and four", and then in the customary terms, as "sixty-four". Information is now given that for the purpose of counting them, objects are grouped or put together in tens, that names are given to these ten-groups, and so the figures already learned can be used to show how many ten-groups as well as ones there are. How this is done remains to be explained.

TWO-DIGIT NOTATION: ILLUSTRATION

Let each pupil have a number of the small sticks, say 34. Have the pupils count these and group them in tens. The teacher may then ask the pupils to give in tens the number of sticks—three tens and four. "In what other way can this be said? Thirty-four."

A pupil may then be required to take the three groups in his left hand and the four single sticks in his right hand and stand close to the black-board facing it, with the objects held above his head. The teacher will then show how 34 is written, namely, 34.

"What number did we say is written in this way? Thirty-four. How many figures does it take to write thirty-four? Two. What are they? 3 and 4. Look again at the sticks. Of what have you three? Tens. Of what have you four? Ones. Then 34 means how many tens and how many ones? Three tens and four ones. Once again, what is the number called? Thirty-four."

Now have the pupils make with sticks 4 tens and 2 ones. "How many sticks have you? Forty-two."

Again, the pupil holds the four groups in his left hand and the two single sticks in his right hand and stands close to the black-board facing it, and the teacher shows how forty-two is written, namely, 42. "What is this number? Forty-two. How many figures are there? Two. What are they? 4 and 2.

"Look at your sticks. Of what have you four? Tens. Of what have you two? Ones. Then 42 means how many tens and how many ones? Four tens and two ones. What name is given to this number? Forty-two."

The pupils will be asked to make with the sticks other bundles of tens and ones, and the work will proceed as before.

After dealing with four or five numbers in this way, the pupils will then be asked to pick out, say, two tens and five ones. "How many sticks have you? Twenty-five. Now, write that on the black-board." (The pupil writes 25.) "How many figures did you use? Point out for what the 2 stands, and also the 5."

This may be followed by similar exercises. The teacher will write on the board in figures some number, say 18, and ask the pupil to make it up with sticks. "How many sticks have you? One ten and eight ones. What is the other name for this? Eighteen. How is eighteen written? 18."

Now have the pupils write 10 in figures. "How many figures are there? Two. For what does the 1 stand? The 0? How then would you write two tens, or twenty? Four tens, or forty? For what does 30 stand? Read 70."

Now have pupils read 33 and ask them which of the 3's is for the tens and which for the ones.

From the foregoing, the pupils should know that, to write each of the numbers from 10 to 99, two figures are used—the one on the right, the *first* figure, being for the ones, and the one in the *second* place being for the tens.

ALTERNATIVE METHOD

The numbers used in the above illustration were selected to show a change of ten as well as a change of unit—34, 42, 25, 18. The method is thus generalized. Equally good results may be obtained by changing the unit without changing the ten, then taking a new ten and proceeding as before. In this method the numbers should be taught, first in an irregular order, say, 34, 36, 32, 39, 31, 30; and then in their regular sequence, 30, 31, 32, 33, 34, etc.

Practice will now be given in counting objects and writing the numbers so that they may be learned in their regular sequence: 1, 2, 3, 4,—96, 97, 98, 99; and the series from 1 to 100 may be properly connected.

APPLICATION

For class work, and afterwards for seat work, the following types of exercises may be given:

1. Write, in figures, all the numbers from 1 to 18, from 19 to 36, from 28 to 40, from 64 to 86, from 1 to 50, etc.

2. Name the numbers: 36, 82, 15, 28, 40, 9, 66, 12, 70, etc.

3. Place in order, from least to greatest, the numbers: 10, 82, 16, 47, 20, 35, 7.

4. What number is next above, or next below, 36, 29, 17, 53?

5. Between what two numbers do we find: 13, 20, 25, 61, 49?

6. Add: 20+5, 10+8, 30+6, 40+1, etc.

7. Add: 10+3+4, 20+5+4, 30+1+6, etc.

8. Add: 2 tens+1 ten, that is, 20+10; add: 4 tens+3 tens, that is, 40+30, etc.

CHAPTER V

ADDITION

ORDER OF EXERCISES

KNOWING the numeration and notation of numbers to 99, and that objects are grouped in tens for the purpose of counting, the pupils will be able to extend the work in addition, which hitherto has been confined to the addition of numbers not exceeding 10 in their sum.

In single-column addition, the work may be graded or arranged in the order following:

1. The addition of numbers, the sum not to exceed 10. For example: 4+2, 3+2+3, 2+3+2+2.

2. The addition of any number less than 10 to any ten. For example: 10+6, 30+7.

3. The addition of any numbers the sum of which does not exceed 10 to any ten. For example: 20+3+2, 40+1+2+4.

4. The addition of tens, as: 6 tens and 2 tens, that is, 60+20.

5. The addition of 10 to any number. For example: 32+10=4 tens and 2=42.

6. The addition of two addends involving the addition of tens to facts already learned. For example: 2+4, 12+4; 14+2, 24+2, 44+2; 5+3, 15+3; 13+5, 23+5, 33+5; etc.

7. The addition of numbers to any number of two digits which will not make the sum exceed the ten above that within which the two-digit number lies. For example: 23+2+3+1, or 11+4+1+2.

8. The addition of any single column of addends which are such that they can be arranged in successive groups, the sum of each group except the last being 10. For example: 3+4+3+5+4+1+8+2+3+4.

9. The addition of any column of single-digit numbers, the exercises to be graded by drilling first on two addends, then three, four, and so on.

SUGGESTIONS

In the foregoing, some two-digit numbers are used, but a little consideration will show that the pupil will think of each of these as if the number were wholly in the units' column. For example, in adding 23+2+3 he thinks of the numbers as 23 units+2 units and says 23 and 2 are 25, 25 and 3 are 28.

In learning to add it is well to have the pupil repeat both the addends and the sum. For example, in adding 3+2+4, he should say 3 and 2 are 5, 5 and 4 are 9. This repetition helps to fix the association between the addends and the result. In drill, however, where rapidity is one of the main objects, the sum only should be named, and the pupil would then say, in the example given, 3, 5, 9.

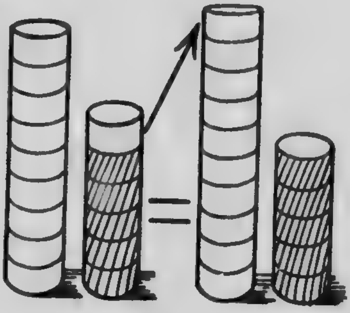
The first eight types of exercises given above are easily illustrated and require little more than a knowledge of the addition facts of the numbers from 1 to 10; but the ninth type requires a knowledge of the addition "facts" of the numbers from 10 to 20, and these must now be taught.

ADDITION FACTS

NUMBERS FROM 10 TO 20

In teaching these facts, the teacher will do well to take the work in easy stages and not to hasten unduly. The method to be employed is precisely that used for the numbers from 1 to 10.

The steps are:



9 + 6 = 15

1. The pupil *discovers,* by using a set of objects, the "fact" to be learned. For example, 15=9+6.

2. He *verifies,* by using several other sets of objects, pictures, etc., the discovered fact, that 9+6=15.

3. He *memorizes,* by repetition and the use of devices, the verified fact, 6+9=15.

4. He *applies* the memorized fact, independently and in combination with other facts, in oral and written exercises and problems.

As previously stated, it must ever be kept in mind that each and every addition fact must be so thoroughly memorized that the pupil can answer without any hesitation each or any of the following questions:

9+6=? 9+?=15 ?+?=15

SYSTEMATIC ORDER

Although all addition facts are of equal importance and must in the end receive equal consideration, yet, that none may be omitted, it will be necessary to deal with them in some systematic order. One or other of the following schemes is suggested:

First Scheme

1. Teach addition facts of 11, and with each fact teach the associated facts which may be obtained from the original facts by the addition of tens. For example:

Facts	Associated Facts		Extensions
5+6	15+6,	25+6,	35+6, etc.
6+5	16+5,	26+5,	36+5, "
4+7	14+7,	24+7,	34+7, "
7+4	17+4,	27+4,	37+4, "
etc.	etc.	etc.	etc.

2. Teach, similarly, the other numbers in their natural sequence: 12, 13, 14, etc.

Alternative Scheme

1. Teach the addition of those pairs of numbers whose sum ends in 0, taking with each pair its associated extensions. For example:

Facts	Extensions	
5+5	15+5,	25+5, etc.
1+9	11+9,	21+9, "
9+1	19+1,	29+1, "
2+8	12+8,	22+8, "
8+2	18+2,	28+2, "
etc.	etc.	etc.

2. Teach, similarly, those pairs of numbers whose sum ends in 9.

3. Teach in regular succession the addition of those pairs of numbers, with associated extensions, whose sum ends in 8, in 7, in 6, in 5, in 4, in 3, in 2, in 1.

By either of these schemes the pupil learns to add any number less than 10 to any number less than 100, and this knowledge is really all that is required for the addition of any set of numbers, no matter how many or how large, for *all addition resolves itself into this one operation or a repetition of it.*

It may be noted that in adding any number which is less than 10, say 5, to any number which is greater than 10 but less than 100, say 36, the result may be obtained by a mental process such as this. 36+5=30+6+5=30+11 =41.

APPLICATION

As each set of addition facts, with its associated extensions, is learned, it should be at once inserted into the oral and written exercises for class and seat work. For instance: 6+5, 16+5, etc., having been taught, these results might then be given in exercises such as:

7	3	2	3
5	5	5	4
5	5	6	5
5	1	2	6
2	9	7	3
3	5	5	6
6	2	4	5
5	4	22	36

Such exercises should, of course, contain old combinations of numbers as well as new, until all possible combinations have been thoroughly grasped and until the pupil can add any single-column sums, no matter what may be the arrangement of the addends.

DRILL: ACCURACY AND RAPIDITY

A great deal of drill will be necessary to teach the combinations thoroughly and to make the pupil add with accuracy, facility—that is, ease, readiness, and confidence—and, finally, with rapidity. The resourceful teacher will seek devices to give variety to his work, to create a spirit of healthy emulation among his pupils, to strengthen the efforts of the weaker ones, and to encourage those who are slow and backward. Every exercise should have some definite purpose—to teach, to drill, or to test.

Accuracy comes from neatness of work, careful memorization of tables, careful grading, much interesting practice, and constant checking of results. Inaccuracy is not infrequently due to mental worry and fatigue. Children are not capable of long-continued concentration and are often appalled by the magnitude of the questions in addition and subtraction set before them.

Rapidity depends chiefly upon accuracy and upon varied and frequent practice. It is greatly aided by oral drill and frequent "time tests", that is, tests requiring accurate results of given exercises done within a prescribed time limit, this limit being made shorter or longer as the necessities of the pupils may demand.

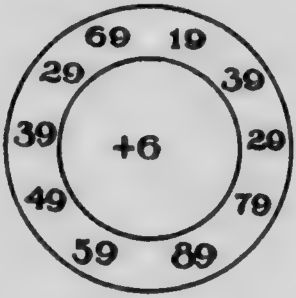
Drill for the purpose of establishing habits of rapid and accurate handling of numbers should be begun as early as possible after school entrance and carried forward with special care at least to the end of the Third Form. Short and frequent periods will be found more effective than the same total time used in long periods.

It will be found generally true that pupils have a tendency to make certain habitual "type-errors". These should be carefully listed by teachers and made the subject of vigorous drill. In most schools much time is squandered in going over matter which the pupils already know, and energy is not concentrated upon the particular difficulties that should be attacked.

Among the more difficult addition combinations may be mentioned, 9+7, 7+8, 6+7, and 9+8.

DEVICES

For rapid oral drill on the addition facts and their combinations, the following devices may be of service. These may be kept permanently on the black-board, ready for use:



0	10	30	40	80	70	60	90	50	20
1	21	11	51	31	61	41	81	91	71
2	12	22	52	62	32	92	72	42	82
3	33	23	63	43	93	73	53	83	13
4	54	34	14	74	24	84	44	64	94
etc			etc			etc		etc	
etc			etc			etc		etc	

The circles explain themselves. The teacher will be able to complete the rectangular device. In this the pupil is asked to add a number less than 10 to each number in any row of numbers which the teacher may select, the additions in the horizontal rows to be made from left to right or from right to left, and in the vertical rows from the top downward or from the bottom upward. If greater variety is deemed necessary, it may be furnished by adding any specified number to any number of numbers on the diagram to which the teacher may point, or by selecting any row of numbers and adding two numbers alternately; for example, take the fifth row and add alternately 8 and 6, the sums are 12, 60, 42, 20, etc.

A further device for drill on combinations may be secured by arranging the digits as follows:

(*a*) 1 2 3 4 5
9 8 7 6 5

(*b*) 1 2 3 4 5
0 9 8 7 6

(*c*) 1 2 3 4 5 6
1 0 9 8 7 6

(*d*) 2 3 4 5 6
1 0 9 8 7

(*e*) 2 3 4 5 6 7
2 1 0 9 8 7

(*f*) 3 4 5 6 7
2 1 0 9 8

(*g*) etc., etc.

These added up and down (without carrying) give all the possible combinations or endings found in addition.

Count by 2's to 20. Count by 2's from 1 to 23, from 7 to 29. Count by 3's to 36. Count by 3's from 1 to 22, from 5 to 38, from 32 to 50. Count by 4's, etc.

Table of combinations of single-digit addends, omitting 1 and 0:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2 2	3 2	4 2	5 2	6 2	7 2	8 2	9 2	9 3	9 4	9 5	9 6	9 7	9 8	9 9
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		3 3	4 3	5 3	6 3	7 3	8 3	8 4	8 5	8 6	8 7	8 8		
		6	7	8	9	10	11	12	13	14	15	16		
				4 4	5 4	6 4	7 4	7 5	7 6	7 7				
				8	9	10	11	12	13	14				
						5 5	6 5	6 6						
						10	11	12						

ADDITION OF TWO COLUMNS

In the addition of two columns there are two principles to be emphasized: These are, (1) that the tens found in the sum of the units' column are carried to the tens' column, and (2) that units are added to units and tens to tens.

In single-column addition, when 7+8+9 is added, the sum is 24 units, or 2 tens and 4 units, which is written by putting the 4 in the units' place and the 2 in the tens' place, that is, the 2 tens are "carried" to the tens' place. It is now to be shown that these 2 tens are added to any other tens found in the tens' column.

Illustration

Aim: To show how to find the sum of two-digit addends.

The steps on which it will be necessary to drill are as follows:

1. Add 3 tens to 2 tens.
2. Add 3 tens to 2 tens and 4.
3. Add 3 tens to 24.
4. Add 30 to 24.
5. Add 30 to 9+7+8.
6. Then what is the sum of 30+9+7+8? of 39+7+8?
7. Add $\begin{array}{r} 8\\ 7\\ \underline{9} \end{array}$ Then add $\begin{array}{r} 8\\ 7\\ \underline{39} \end{array}$
8. Where did the 5 tens come from? Where did the 2 tens come from?
9. Then add $\begin{array}{r} 6\\ 5\\ \underline{47} \end{array}$ $\begin{array}{r} 8\\ 4\\ \underline{29} \end{array}$ $\begin{array}{r} 7\\ 6\\ \underline{18} \end{array}$
10. If 1 ten is added to this last, what is the sum? If 2 tens are added?
11. Then add $\begin{array}{r} 7\\ 26\\ \underline{18} \end{array}$

    Where did the 5 tens come from? Where did the other 2 tens come from?
12. Then add $\begin{array}{r} 45\\ \underline{28} \end{array}$ $\begin{array}{r} 68\\ \underline{29} \end{array}$ $\begin{array}{r} 37\\ \underline{27} \end{array}$ $\begin{array}{r} 29\\ 15\\ \underline{42} \end{array}$ etc.

If necessary, each or any of the above steps may be verified or explained objectively by using the units' and tens' material.

In the foregoing, the two-column addition has been developed out of the single column. The double column might have been taken at once and the process explained objectively.

Illustration

Of the four types of exercises,

$$(1)\ \begin{array}{r}52\\ \underline{37}\end{array}\qquad (2)\ \begin{array}{r}64\\ \underline{75}\end{array}\qquad (3)\ \begin{array}{r}29\\ \underline{38}\end{array}\qquad (4)\ \begin{array}{r}48\\ \underline{95}\end{array}$$

type (3) is selected for illustration.

The teacher will give each pupil the necessary splints or cylindrical blocks and ask each to pick out 29. This will be done by taking out 2 tens and 9 ones. Now ask for 38, which will be shown as 3 tens and 8 ones. The pupils will next place the two numbers together, that is, add them, and express the result as 5 tens and 17 ones; and this as 5 tens, and 1 ten and 7 ones; and this as 6 tens and 7 ones, or 67.

The teacher will then ask a pupil to write on the board the numbers to be added and their sum thus:

$$\begin{array}{r}29\\ \underline{38}\\ 67\end{array}$$

The pupils will be questioned as to how they put the splints together to get the answer 67, and then questioned so as to lead them to see that the numbers represented on the board may be put together in the same way as the splints; thus, the 3 tens and the 2 tens, and the 8 ones and the 9 ones; the 17 ones can be changed into 1 ten and 7 ones, and the 7 ones written down; the 1 ten is added to the 3 tens and the 2 tens, and the sum, 6 tens, is written down, and the answer is 67.

This will be followed by a few similar examples, such as,

$$\begin{array}{r} 64 \\ \underline{17} \end{array} \qquad \begin{array}{r} 56 \\ \underline{28} \end{array} \qquad \begin{array}{r} 27 \\ \underline{45} \end{array} \text{ etc.,}$$

to be worked out with the splints.

The pupil may now be asked to find the sum of 25 and 69 by using the figures only and use the splints to verify the result. He may then discard the splints and add by using the figures only. The number of addends may be increased to three, four, etc., as the pupil acquires skill. Three columns should now offer little difficulty.

REPEATED ADDENDS AND CHECKS

With two-column addition, drill should be given in the additions of 1's, 2's, 3's, etc., and the results should be memorized, in order to build up what later will be found to be the multiplication table.

All answers should be checked. For addition, the usual check is to add from the bottom upward and then from the top downward.

CHAPTER VI

NUMERATION AND NOTATION

HUNDREDS, ETC.

WHEN the pupil meets with such an exercise as 63+84, or 37+98, he finds in his answer an extra figure or a *new left-hand* place, and so he has to learn the use, value, and name of this place. In other words, he has to learn to read and write the three-digit numbers.

He can easily be led to see that the sums 147 and 135 are respectively 14 tens and 7, and 13 tens and 5.

He may then be told that as we group *ones* together in bundles of tens, so we group the tens in bundles of *ten-tens*. He does this with his counting blocks or splints and sees that 147 will give 1 bundle of ten-tens, 4 bundles of tens, and 7 ones.

He is then told that a bundle of ten-tens has a name and is called *one hundred.* As he already knows how to read 47 he can read the whole number 147. He learns also that the *third* figure is for the hundreds and that it is placed at the left of the tens.

He is now given practice, with splints, in grouping ones into tens, and tens into ten-tens, or hundreds. He reads the results of his grouping as hundreds, tens, and ones, thus: 4 hundreds, 3 tens, and 6, that is, four hundred and thirty-six. He then writes the number as 436 and points out the figure which represents the hundred bundle, the ten bundle, and the ones, respectively.

After some practice of this kind, the teacher writes some number on the black-board, say 251, and asks each pupil to represent it with his splints and then read it. This should be followed by representing with splints such a number as 3 hundred and 5, the pupil being taught to indicate the absence of the ten by placing a cipher between the hundreds and the ones, thus 305. Practice in reading and writing such numbers should now be given.

The pupil next represents with splints numbers such as 2 hundreds and 4 tens in which the *ones* are missing. He will be taught to show, by means of the cipher, the absence of the *ones* and to write the number thus, 240. He can now read, write, and interpret all such numbers.

Finally, he is asked to express with splints the hundreds alone when both tens and ones are missing. The writing and reading of these offer little difficulty, and the pupil is ready for drill exercises.

Along with the foregoing exercise, the pupil should be given such practice in counting as will enable him:

1. To count from any number below any hundred to any other number below that same hundred;

2. To count from any number below any hundred to any number below the next succeeding hundred;

3. To count, if necessary, from any number to any other number up to one thousand;

4. To name the number immediately following or immediately preceding any named number;

5. To arrange in their proper number-sequence from the least to the greatest or from the greatest to the least, any number of given numbers.

THOUSANDS, ETC.

Illustrate concretely 999. Add 1 unit—block or splint—and note the changes. What are they? The 1 unit with the 9 units will make 1 bundle of tens. This with the 9 bundles of tens will make 1 ten-ten, or hundred bundle, and this with the 9 bundles of hundreds will make 1 ten-hundred bundle, which gives a new left-hand position requiring a new name—*thousands*. Thus the pupil is introduced to the four-digit number and is prepared to read and write it.

OTHER DIGITS

For additional figures all that remains for the teacher to do is to explain the device used in naming the places in which the figures are written. This consists in grouping the places, every three places from the right making a group. These groups are named units, thousands, millions, billions, etc. Each of the three places in the group is also named, the same place in each group having the same name, units, tens, hundreds.

CHAPTER VII

SUBTRACTION

PREPARATORY WORK

SUBTRACTION should be taken up before addition is completed, and the teacher who, in teaching addition, keeps subtraction in view, will experience but little difficulty when it becomes necessary to deal formally with this topic. The two processes are so closely related that some advocate the teaching of them together, and although it is perhaps better to emphasize but one of them at a time, yet there should be given in addition many exercises, the aim of which is to prepare for subtraction.

For instance, the pupil will be taught to add by 2's, 3's, 4's, etc. He may, at the same time, be taught to count *backward* by these same numbers and so learn to subtract any number less than ten from any number less than one hundred. Again, on the addition tables there are three types of questions which the pupil must answer readily. They are (1) $6+8=?$, (2) $6+?=14$, and (3) $?+?=14$. In drilling on the second of these types, $6+?=14$, *oral* problems involving subtraction might be given, such as:

A boy needs 14 cents. He has now 6 cents. How many cents has he still to get? Then, what is the difference between 14 and 6?

John bought 14 oranges. He gives away 6; how many has he left? Then, what is the *remainder* when 6 is taken (that is, *subtracted*) from 14?

So, too, when any two addends are given, the pupil may be asked: (1) Of what two parts is the sum made up? (2) If one of these parts is taken away, or *subtracted*, from that sum, what is left? Then, by a series of graded exercises, he may easily be led to find *one* of two numbers which make up a stated sum, the other of the two numbers being given.

Children like to work at puzzles, and the problems here referred to may, at first, be assigned as such. For instance, the teacher may give additions of two addends such as:

$$\begin{array}{r} 364 \\ 158 \\ \hline ??? \end{array} \qquad \begin{array}{r} 454 \\ 294 \\ \hline ??? \end{array} \qquad \begin{array}{r} 637 \\ 349 \\ \hline ??? \end{array}$$

and then ask the pupils to try to find the number which must be added to another so as to make a given sum; or, in other words, to replace the question marks in the following by the proper figures, so that the two numbers above, when added, will give the number below the line:

$$\begin{array}{r} 425 \\ ??? \\ \hline 867 \end{array} \qquad \begin{array}{r} 369 \\ ??? \\ \hline 583 \end{array} \qquad \begin{array}{r} 275 \\ ??? \\ \hline 642 \end{array}$$

It will then be an easy matter to write one of the numbers *below the sum*, draw a line, and ask the pupils to find the other number and place it *below the line*, in order that the sum may be made up of the two parts which are separated by the line, thus:

$$\begin{array}{r} 534 \\ 276 \\ \hline ??? \end{array}$$

The pupil can then answer orally the question: 276+?=534; and also the question: If 276 is taken away, or subtracted from 534, what is the remainder?

He will thus discover for himself the method of subtraction known as the *Additive Method* or the *Computer's Method.*

ADDITIVE METHOD

This method depends upon the principle that the sum of the remainder and the subtrahend must always equal the minuend. It requires no tables except those already learned in addition, and it should lead to accurate results, since every part of the answer is *checked.* It is the method used by business men in making change.

It should be noted, however, that in this method there are two parts to the solution of subtraction problems. The first is *to find the two addends into which the minuend is divided,* and the second is *to find the remainder when one of these addends is taken away.* The first is all that is shown in the *written* work, the second part being given *orally.*

The pupil soon discovers that the *addend* to be found is the same as the *remainder* or difference required, yet the teacher must give emphasis to the *two* questions: "What is the addend?" and "What, then, is the remainder?"

ORDER OF STEPS AND EXERCISES

The order to be observed in grading the exercises may be given as follows:

1. Review of addition tables orally, drilling especially on such questions as: (*a*) "7+?=15", and (*b*) "Take, that is *subtract,* 7 from 15, and what remains?"

2. Give pairs of numbers whose sum is 8, 11, 17, etc., one number of the pair being given and placed above a line, the other to be found and placed below the line. The addition facts of 8 are:

$$\begin{array}{ccccc} 6 & 3 & 5 & 1 & 4 \\ \hline ? & ? & ? & ? & ? \end{array}$$

The addition facts of 11 are:

$$\begin{array}{ccccc} 5 & 9 & 2 & 4 & \\ \hline ? & ? & ? & ? & \text{etc.} \end{array}$$

These numbers being found, the question will then be asked in this form: If 6, 3, 5, 1, etc., is taken from 8, what remains? If 5, 9, 2, 4, etc., is taken from 11, what remains? Etc.

3. Change the form of 2, above, by writing the numbers, of which the addition facts are to be given, above those facts, thus:

$$\begin{array}{cccc} 8 & 8 & 8 & \\ 6 & 3 & 5 & \text{etc.} \\ \hline ? & ? & ? & \end{array} \qquad \begin{array}{cccc} 11 & 11 & 11 & \\ 5 & 9 & 2 & \text{etc.} \\ \hline ? & ? & ? & \end{array}$$

$$\begin{array}{cccccc} 14 & 10 & 9 & 17 & 15 & \\ 5 & 6 & 7 & 8 & 12 & \text{etc.} \\ \hline ? & ? & ? & ? & ? & \end{array}$$

Again, in each example ask the question: "What is the remainder?"

4. The meaning of the words *subtract* and *remainder* should be made clear, and the pupils should be told that when they were adding numbers they were said

to be doing *addition;* now, when they are subtracting one number from another, they are said to be doing *subtraction.* The words, *subtrahend* and *minuend,* may be introduced whenever the teacher thinks they can be used and understood by the pupils.

5. Give questions requiring the subtraction of any number less than 10 from any number less than 100, observing at first a sequence suggested by the endings of the minuend; later, miscellaneous questions, thus:

$$\begin{array}{r}12\\5\\\hline ?\end{array}\quad\begin{array}{r}32\\5\\\hline ?\end{array}\quad\begin{array}{r}62\\5\\\hline ?\end{array}\ \text{etc.}\qquad\begin{array}{r}15\\7\\\hline ?\end{array}\quad\begin{array}{r}25\\7\\\hline ?\end{array}\quad\begin{array}{r}45\\7\\\hline ?\end{array}\ \text{etc.}$$

$$\text{then}\quad\begin{array}{r}22\\6\\\hline ?\end{array}\quad\begin{array}{r}40\\9\\\hline ?\end{array}\quad\begin{array}{r}34\\8\\\hline ?\end{array}\ \text{etc.}$$

6. Give questions requiring the subtraction of tens from other tens, thus:

$$\begin{array}{r}40\\10\\\hline ??\end{array}\quad\begin{array}{r}50\\30\\\hline ??\end{array}\ \text{etc.}$$

7. Give such questions as:

$$\begin{array}{r}54\\31\\\hline ??\end{array}\quad\begin{array}{r}45\\12\\\hline ??\end{array}\quad\begin{array}{r}68\\25\\\hline ??\end{array}$$

8. Give such questions as:

$$\begin{array}{r}43\\29\\\hline ??\end{array}\quad\begin{array}{r}35\\19\\\hline ??\end{array}\quad\begin{array}{r}70\\24\\\hline ??\end{array}$$

9. Give questions of which the following are types:

$$\begin{array}{r} 217 \\ 5 \\ \hline ??? \end{array} \qquad \begin{array}{r} 302 \\ 8 \\ \hline ??? \end{array} \qquad \begin{array}{r} 465 \\ 32 \\ \hline ??? \end{array} \qquad \begin{array}{r} 563 \\ 38 \\ \hline ??? \end{array} \qquad \begin{array}{r} 619 \\ 65 \\ \hline ??? \end{array} \qquad \begin{array}{r} 134 \\ 98 \\ \hline ??? \end{array}$$

$$\begin{array}{r} 437 \\ 130 \\ \hline ??? \end{array} \qquad \begin{array}{r} 245 \\ 128 \\ \hline ??? \end{array} \qquad \begin{array}{r} 569 \\ 283 \\ \hline ??? \end{array} \qquad \begin{array}{r} 720 \\ 457 \\ \hline ??? \end{array}$$

10. Miscellaneous questions without reference to order or grading.

Process Illustrated

The process in all these types of problems is essentially the same and may be illustrated by means of such an example as: Subtract 3687 from 5243.

The first part of the solution is to find what number must be added to 3687 so that the sum may be 5243.

To find this, the pupil sets the numbers down thus:

$$\begin{array}{cccc} 5 & 2 & 4 & 3 \\ 3 & 6 & 8 & 7 \\ \hline \end{array}$$

He tries to find out what figures must be placed under the 7, 8, 6, and 3, in order that the number found when added to 3687 will give 5243.

Now the only figure that can be put in any one place is 1, or 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, or 0. He tries these under the 7 and he finds that 6 is the number required to be added to 7 to give a sum ending in 3, and that the three is the unit of 13, that is, of the sum of the 6 and the 7. He then adds orally the 7 and the 6 and gets 13. There is 1 to carry on to the 8, making 9; he again tries the number 1, or 2, or 3, etc., for the tens' place and finds that 5 is the number required to be added to 9 to give a sum ending in 4, and that the 4 is the ending of 14. He now adds orally the 5 to the 9, gets 14 tens, and finds that there is 1 hundred to be carried to the 6 hundreds, making 7 hundreds. He once more tries in the third place the numbers 1, 2, 3, etc., and finds 5 to be the one required. This added to the 7 makes 12 hundreds, and there is 1 thousand to be carried to the 3 thousands, making 4 thousands, and the figure required in the fourth place to make 5 thousand is, therefore, 1.

His work now appears thus:

$$\begin{array}{cccc} 5 & 2 & 4 & 3 \\ 3 & 6 & 8 & 7 \\ \hline 1 & 5 & 5 & 6 \end{array}$$

He now knows that if 1556 is added to 3687 the sum will be 5243, and, consequently, if 3687 is subtracted from 5243, the remainder will be 1556.

After a little practice, the correct figure in the remainder is readily found. The discovery is made that, when any digit in the minuend is less than the corresponding digit in the subtrahend, the number to be made up is 10 more than that represented by the minuend digit. For example: 3 is less than 7, so there is to be added to 7 a number which with 7 will make 13. Why not 23, or 33?

CHECKING

The subtraction checks usually employed are:

1. Add subtrahend and remainder and the result will be the minuend.

2. Subtract the remainder from minuend and the result will be the subtrahend.

DECOMPOSITION METHOD

A second method employed in subtraction is known as the *Decomposition Method.* It is one which, perhaps, finds the most favour with teachers, owing no doubt to the fact that it lends itself readily to concrete illustration and grows out of the "carrying" process in addition.

As a basis for this method, the pupils must, out of the addition table, form a subtraction table and be able without hesitation to give the result when any number less than 10 is subtracted from any number less than 19. For drill on this part of the work, devices similar to those suggested for addition tables, may be used.

The grading of exercises may follow much the same order as that suggested for the Additive Method.

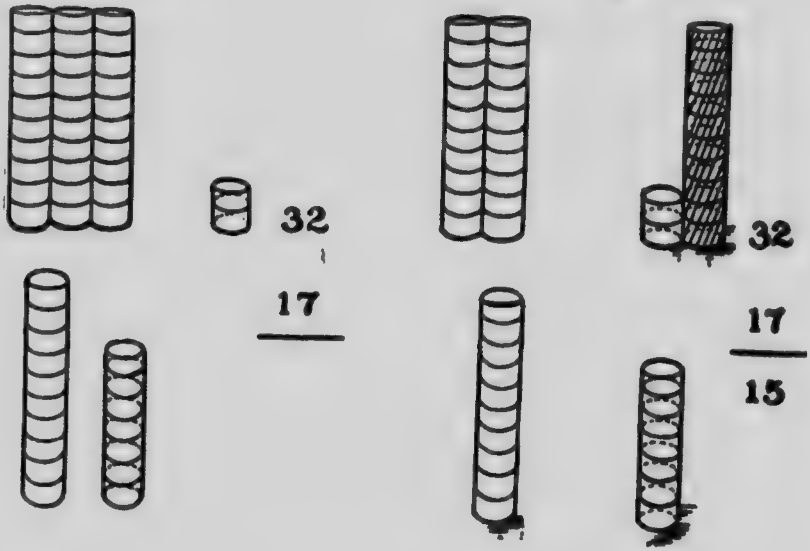
The only difficulty which has to be overcome in subtraction, no matter what the method employed may be, is to teach the pupil to subtract where any digit in the subtrahend is greater than the corresponding digit in the minuend.

Illustration

This may be illustrated thus: Subtract 17 from 32. The figures are set down thus:

$$\begin{array}{r} 3\ 2 \\ \underline{1\ 7} \end{array}$$

At once the pupil meets the difficulty of subtracting 7 units from 2 units. He is asked to represent 32 with his counting blocks. This is done with 3 tens and 2 ones. Then the 3 tens and 2 ones are changed or decomposed into 2 tens and 10 ones and 2 ones, or into 2 tens and 12 ones, from which 1 ten and 7 ones is easily taken, leaving a remainder of 1 ten and 5 ones, or 15. The counting blocks or splints will show the operation, thus:


Step I
Step II

Further Illustration

The Decomposition Method may be further illustrated thus: Subtract 2359 from 8143.

8 1 4 3=8 1 3 0+13=8 0 0 0+1 3 0+13
2 3 5 9=2 3 5 0+ 9=2 3 0 0+ 5 0+ 9

=7 0 0 0+10 0 0+13 0+13
=2 0 0 0+ 3 0 0+ 5 0+ 9
5 0 0 0+ 7 0 0+ 8 0+ 4=5 7 8 4.

CHAPTER VIII

MEASURES

Early in the child's school experience, the simpler measures should be begun, in order that he may acquire objectively a knowledge of some of the units used.

I. MONEY

The first units encountered are likely to be those connected with money.

The use of the coins may be made a topic for a very interesting language lesson. The different Canadian coins should be shown to the pupils, their values explained, and practice given in changing each, in as many ways as possible, into others of lower denomination. This cannot, of course, be taken in one lesson. It should be distributed according to the pupils' number attainments.

II. LINEAR UNITS—FOOT, INCH, YARD

THE FOOT

To introduce the linear units, the pupils may be asked to suggest different ways in which they can determine how far it is from one side of the class-room to the other. Among other methods, it will probably be mentioned that the distance can be measured by using a stick, or a string, or by stepping it off. Which of these will be the best way, and why? This question will impress the fact that since some ways are better than others we should try to get the best way possible.

If stepping off the distance has been suggested, a small boy might be asked to do this, and the class asked to count his steps. Then a large boy does the stepping and the class counts the steps. How many steps did the first boy take? How many did the second take? Then how wide is the room?

Now give to each of the two boys a stick and let the sticks be the same length. Again ask each to measure the width of the room with his stick and have the class count.

How many times did the first boy use his stick? How many times did the second use his? What then is the width of the room? Why did both boys get the same answer when they used the sticks, but different answers when they used their steps?

We all get the same answer when we measure the same distance or length, if we all use the same length of stick, or step, or string with which to do the measuring.

The teacher may now give each of the pupils a strip of cardboard, or cover paper, or plain wooden ruler (without marking), each one foot long, and give the information that the *length* of this strip, or ruler, is one of the lengths which everybody uses in measuring other lengths or distances. The pupils may then compare their strips with those of other pupils to see that all are of the same length. The teacher will then say that this length is given a name and is called a *foot,* and anything that is as long as one of these strips is *one foot* long; if as long as two of them, it is two feet long, etc.

This will be followed by practice (1) in measuring various lengths, such as the length of the teacher's desk, of the black-board, of the window-sill, the distance of the window from the floor, the width of the door, the height of pupils, etc., and (2) in marking off various distances, such as 4 feet, 6 feet, 9 feet, etc., and in estimating by the eye certain lengths. Lastly, with the foot measure, the pupils can now find the width of the class-room and also its length.

THE INCH

The pupils will endeavour to find with the foot measure the length of their pencils, the width of their books, hands, etc. They will see that these lengths cannot be measured exactly with the foot measure. Why? A smaller measure is needed. The teacher then shows the smaller measure used, say, a narrow piece of cardboard one inch long, and tells them that this length has a name, and is called an *inch,* and that anything that is as long as this is said to be *one inch* long; if as long as two of these, it is two inches long, etc.

Practice with the inch, similar to that given with the foot, will follow, and lengths will also be expressed in feet and inches. The pupils will mark off lengths, such as one foot and two inches, two feet and five inches, and will also find in feet and inches lengths measured in inches, and vice versa.

Now have the pupils find the number of inches in one foot, and let them mark the inch divisions on the foot strip. They may then compare this foot strip with their school ruler. They will see that the divisions on the one coincide with those on the other. They will then be shown that when the ruler is so marked it can be used to measure both feet and inches. Practice in measuring with the school ruler may then be given.

THE YARD

The length of the black-board is to be found. Ask the pupils which measure should be used, the inch or the foot. Why? If then we wish to measure something longer still, what kind of measure will be even better than the foot? A longer one. Now show the pupils the longer measure which is used, and tell them that this length is called a *yard*. Give practice in the use of this measure by marking off a certain number of yards; a certain number of yards and feet; a certain number of yards, feet, and inches. Now have them measure a certain number of feet or of inches, and then by measuring express these in yards, feet, and inches.

The yard may be measured by feet and by inches and the divisions marked. The pupils will then see that the yard measure, when so divided, can be used for measuring in yards, or feet, or inches.

III. OTHER UNITS

In a similar manner, the other units—pint, quart, pound, ounce, day, week, peck, bushel, etc., may be taken up in their proper places in the course of studies. All tables should be taught as objectively as possible. Several of the standard measures form a part of the necessary equipment of every Public School, and the teacher should see that the pupils have a definite notion of every measure taught. The length of a rod (16½ ft.) should be marked off on the black-board or floor; points one mile distant from the school should be mentioned; the pupils should also know from actual experience what is meant by one square inch, one square yard, one square rod, one acre, one cord, one ton, one gross, etc.

When any unit of measure has been taught, it should be given its place among the others of the same table, and the pupil should be able (1) to name these units in the proper order from smallest to largest and from largest to smallest, thus: inch, foot, yard, rod, mile; mile, rod, yard, foot, inch; and (2) to give the numerical relation between them, as 12 inches = 1 foot, etc.

Finally, the pupil should be able to mention specific articles, the quantity of which is measured by given units. Thus cloth is measured by the yard, grain by the bushel, hay by the ton, tea by the pound, eggs by the dozen, land by the acre, etc. Current prices of these articles should also be quoted where possible.

CHAPTER IX

MULTIPLICATION AND ROMAN NOTATION

THE TABLES

THE groundwork for the multiplication tables is, of course, the mastery of addition. The pupil will incidentally learn portions of these tables, when he is taught to add by 2's, 3's, 4's, etc.; and most pupils will know at least the two-times and ten-times tables long before they reach formal multiplication. As the key to multiplication and division, the tables must be learned and learned perfectly—so thoroughly that any pair of factors will instantly suggest the product. "There must be no halting memory summoning attention and judgment to its aid."

Multiplication should commence with a knowledge of but one table or even less, for this will show the use of the tables and furnish an incentive to learn them. The pupils must be taught to construct for themselves all the tables. This they will do at first by means of addition, but as time goes on, they should be led to discover certain relations between numbers and their products, which will aid them to secure results more rapidly.

RELATION BETWEEN TABLES

Although, as stated above, the pupils will probably know the two-times table, nevertheless the formal study of it will be useful in the study of other tables and in leading them to believe that they are about to take up a topic which is easily mastered. This table, when learned, should be applied immediately, and the pupil set to work to construct the three-times table. This may be done through addition, but the pupil may also be shown how to construct it from the two-times table. For instance, he sees that 2 times 7 is 14, and that 3 times 7 is 21, that is, 7 more than 2 times 7. So also 3 times 9 is 9 more than 2 times 9. He will likewise see how to derive the product of any two factors in the table from the product of the pair immediately preceding. Knowing that 3 oranges, at 5 cents each, will cost 15 cents, he will know that 3 oranges, at 6 cents each, will cost 18 cents; for each of the latter costs *one* cent more than *each* of the former, and the 3 oranges will cost a total of 3 cents more than at the former price.

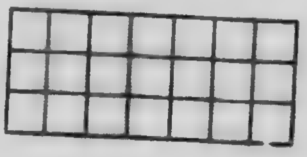
$$\begin{array}{r} 7 \\ 7 \\ \hline 14 \end{array} \qquad \begin{array}{r} 7 \\ 7 \\ 7 \\ \hline 21 \end{array}$$

So, too, if the pupils know that the cost of 3 pencils, at 4 cents each, is 12 cents, they will know that the cost of 3 pencils at 8 cents each, will be twice 12 cents, that is, will be 24 cents. Thus they will have various ways of constructing, and of recovering, when momentarily forgotten, the product of any pair of numbers.

LAW OF COMMUTATION

As an important aid in learning the tables, the teacher must, from the first, lead the pupils to see that the product of any two numbers is the same, no matter in which order they are taken.

For instance, 3 times 7 is equal to 7 times 3. This will be made clear by the accompanying diagram, in which there are three horizontal rows of 7 squares each, or 7 vertical rows of 3 squares each.



This law materially lessens the labour required in constructing and memorizing the tables. For, when one table is known, part of every other table is also known. Thus, if the pupil knows the six-times table, he also knows the remaining tables up to six; that is, he knows 7 times up to 7 times 6, 8 times up to 8 times 6, 9 times up to 9 times 6, etc.

VALUABLE HINTS

The following may be noted:

1. The products in the five-times table end either in zero or 5, according as the other factor is even or odd.
2. In the nine-times table, in every product up to 9 times 10, the sum of the digits is 9 and the tens' digit is always one less than the number multiplied by 9.
3. In the tables for the even numbers, the products are all even; while in the tables for the odd numbers, the products are alternately odd and even according as the number to be multiplied is odd or even.
4. The simplicity of the ten-times and eleven-times table is at once apparent.

Oral drill on the tables can best be conducted during class recitation and by the teacher. If a pupil misses a table or part of a table, show him how to construct the table or to find the product needed; then have it repeated until he can give, without conscious effort, the product the instant the factors are mentioned.

It is not advisable to learn the tables as tables. The frequent repetition of a table, as a table, makes the saying of it as a whole habitual, so that frequently a pupil cannot give any particular part without starting at the beginning of the table.

Each product in the table should be known independently of every other product, and *this can never be secured if the facts are not memorized separately.* Moreover, 4 and 9 should lead automatically to the product 36, whether one thinks of 4 or 9 first; similarly, 36 should at once suggest the factors 4 and 9, 12 and 3, 6 and 6. Such memorizing is best done by constant and frequent practice with each item, or a few items at a time, in working quickly varied oral problems. Each statement should be grasped in all its aspects and used in many different ways. For example, the following exercises on 4×9=36 might be given:

Four boys had 9 nuts each, how many altogether?

Divide 36 apples equally among 9 boys.

Find the value of 9 pencils at 4 cents each.

Find the value of 4 houses worth nine thousand dollars each.

Find the cost of 9 warships at 4 million dollars each.

Nine baskets of equal size contain a total of 36 dozen eggs; find the number of dozens in each basket.

Find the price of 9 teams of horses at four hundred dollars for each team.

DRILL DEVICES

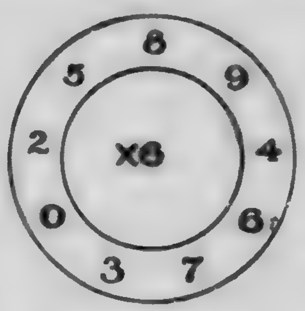
The following devices may also be used to give variety to the drill:

1. A row of numbers is written on the blackboard, and a number written below; thus:

3 9 4 7 6 5 8 1 2

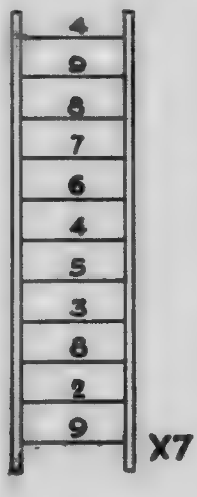
4

The teacher points to any of the numbers in the row, and the pupil states the product of it and the isolated number. When the teacher places the pointer on figure 5, the pupil names the product 20. Speed contests may be inaugurated by means of this device, and competition started among the pupils.



2. "The Circle"

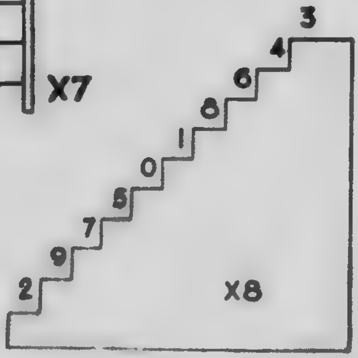
Multiply each of the outer numbers taken in any desired order by the number at the centre. Tell the products only.



3. "The Stairs" or "The Ladder"

(1) Multiply the number on the step to which the teacher points by the number at the side.

(2) Begin at the top step and come down; name the products only.



(3) Begin at the bottom step and go up, skipping every second step.

4. "The Square"

4	9	6	8	2	5	12	3	6
8	7	2	9	6	4	5	7	3
3	4	1	12	6	9	0	2	1
2	8	3	6	4	3	7	12	8
8	9	10	3	11	6	4	9	6
11	2	8	2	5	7	11	6	0
5	4	9	4	2	6	3	7	10
9	8	6	4	9	1	10	5	4
6	5	3	7	8	9	3	0	7

Multiply each number in any row, up, down, right, or left, by a given number.

5. (1) Repeat the table forward.
   (2) Repeat it, missing every other line.
   (3) Repeat the table backward.
   (4) Repeat it backward, missing every other line.
   (5) The teacher names the product, the pupil names the co-factors.
   (6) The teacher names the product and one factor, the pupil names the other.

APPLICATION: WRITTEN WORK AND SEAT WORK

Illustration

As soon as any table, say 3 times, has been learned, the pupil may be taught to apply it.

Take, first of all, a question in addition, such as:

$$\begin{array}{r} 685 \\ 685 \\ 685 \\ \hline 2055 \end{array}$$

Ask the pupil what he sees in this question that is not seen in most questions in addition. The three addends are the same. Then how often is 685 used? Three times. Hence the sum is how many times 685?

Look now at the units' column. What is found in it? Three fives. When added what is the sum of 3 fives? From the three-times table what is 3 times 5? There are two ways then by which to get the sum of the units' column. What are they? What is done with the sum, 15? 5 is placed in the units' column and 1 "carried" to the tens' column.

Look now at the tens' column. What is found in it? Three eights. When added what is their sum? 24. How else do we know that the sum of the three digits is 24? What else is to be added into this column besides the 3 times 8? What is done with the 25 tens? Deal with the hundreds' column in the same way.

In some such manner as this the pupil is led to see the meaning of 3 times 685 and to discover that the sum can be found not only by addition but also by taking 3 times the number which is repeated in the units' column, then 3 times the number which is repeated in the tens' column, and so on, the "carrying" to be performed as in addition. He is then told that this method for finding the sum is called *multiplication,* which is thus seen to be a short method for adding equal addends. He is then shown the form in which the work is set down:

$$\begin{array}{r}685\\ 3\\ \hline 2055\end{array}$$

He once more states how the answer is obtained by the use of the three-times table. Three times 5 units is 15 units. Set down the 5 and "carry" 1 ten to the tens' column. Three times 8 tens is 24 tens, which, with the 1 ten "carried", will make 25 tens. Set down the 5 and "carry" the 2 to the hundreds' column, etc.

The pupil will also be able to state to what the 685 corresponds in the addition question. Where does the 3 come from in the addition question? To what does the 2055 correspond? The *multiplication* names of these numbers may then be given.

SEAT WORK

When the pupil has been taught to multiply numbers of two or three digits by the numbers 2 and 3, he may be assigned seat work of the same nature. The multiplicand should never have more than three figures at first, and need not at any time, before the pupil is promoted to Form III, have more than six figures.

The following examples are suitable for seat work for beginners in written multiplication:

Multiply 687 by 2
789 by 2
863 by 2
654 by 3
987 by 3
999 by 3

Gradually, as the tables are learned, 4, 5, etc., are introduced as multipliers in the problems for seat work. Care should be taken not to assign written multiplication for seat work before written multiplication has been taught in the class, nor should an attempt be made to teach written multiplication before the corresponding tables have been thoroughly learned; and the teacher should remember that the tables can be taught most effectively and thoroughly during the recitation period.

ORDER FOR TEACHING MULTIPLICATION

1. Multiplication by the numbers from 1 to 12 inclusive.
2. Multiplication by 20, 30, 40, etc., and by 200, 300, 400, etc.
3. Multiplication by any two or more digits.

First Topic

The first of these topics has already been explained, but it is to be pointed out that in multiplying by 10 the product is found not by multiplying by 0 and then by 1, but by multiplying by 10 as if it were a single-digit number. The pupil will quickly learn how to write or name the product where any number is multiplied by 10, and should be given oral exercises thereon. These exercises will emphasize the use of the zero, which is to permit the significant digits to be written in their proper places.

Second Topic

Before taking up the second topic—that of multiplication by the exact tens—it will be necessary to lead the pupil to see that if one number is multiplied by another, the product is the same as if the first number were multiplied by a factor of the multiplier and the result multiplied by the remaining factor of the multiplier.

This may be done by asking the pupil to multiply, say 827 by 6, then ask him to multiply 827 by 3 and the result by 2, and compare the final products. Again, multiply 493 by 10, then multiply 493 by 2 and the result by 5, and compare the final results. Now let the pupil suggest two ways in which he could find the product of 549×12. Have him test his suggestions.

Next ask him to suggest a way by which he could find the product of 352 by 15, when he does not know a 15-times table. He will probably advise multiplying 352 by 5 and the result by 3. Let him do this and test his answer by addition—the only other method as yet possible for him to employ.

Now ask him to suggest how he could find the product of 748 by 20. He will probably advise multiplying by 4 and then by 5, or by 2 and then by 10. Let him employ both pairs of factors and he will be prepared to believe his answer correct, since it is the same in both cases. He may now be assured that he has the correct product. He may next state which of the two pairs of factors he would prefer to use, 4 and 5, or 2 and 10. Have him give a reason for his choice.

How then, could we multiply by 30? Give examples such as 426×30; 819×30; etc.

The work will at first be set down thus:

$$\begin{array}{r} 426 \\ 3 \\ \hline 1278 \\ 10 \\ \hline 12780 \end{array}$$

It is now a very simple matter to show how the product might have been obtained by adding a zero to the product of 426×3, that is, by making this product *tens.* The work will be set down thus:

$$\begin{array}{r}426\\30\\\hline 12780\end{array}$$

Multiplication by the exact hundreds will be similarly explained.

Here it will be found helpful to call attention, by review, to the fact learned in notation—that the *place* of a figure gives it a value in units, tens, hundreds, etc.

The following type of exercise may be used:

Read the underlined parts of 642376, 40317, 850216, 372164, 168037, etc.

Third Topic: Multiplication by any two or [illegible] *Digits*

As a basis for this topic it will be nece[illegible] lead the pupils to discover the principle un[illegible]—namely, that a times a number + b times that number is equal to $(a + b)$ times the number.

This can be shown by taking a question in addition where the addends are equal. It will be readily understood that if there are, say, 12 equal addends, the sum may be found by multiplying the addend by 12. The sum may also be found by dividing the question into two parts having, say, 5 addends in one part and 7 in the other. The total sum would be found by adding together the sums of the two parts. But these sums may be found by multiplying the addend by 5 for one part and by 7 for the other. That is, 5 times the addend + 7 times the addend is equal to 12 times the addend.

To further emphasize the principle give such examples as:

1. (*a*) Multiply 729 by 8. (*b*) Multiply 729 by 5, and 729 by 3, and add the products. Compare the answer for (*a*) with that for (*b*).

2. (*a*) Multiply 635 by 20. (*b*) Multiply 635 by 8 and 635 by 12, and add the products. Compare the answer for (*a*) with that for (*b*).

3. From what has been done in examples 1 and 2, state two ways by which we can find the product of 982 × 7; the product of 457 × 11, etc. Test your answer in each case.

4. How now may we find the product of 496 × 24? Do this. What other method have we used to find this product? (Multiply by 4 and the product by 6.) Do this, and see if the answer is the same as that obtained by the first way.

To multiply 496 by 24 by the principle established, the pupil would at first set down the work thus:

$$\begin{array}{r} 496 \\ \underline{4} \\ 1984 \end{array} \qquad\qquad \begin{array}{r} 496 \\ \underline{20} \\ 9920 \end{array} \qquad\qquad \begin{array}{r} 1984 \\ \underline{9920} \\ 11904 \end{array}$$

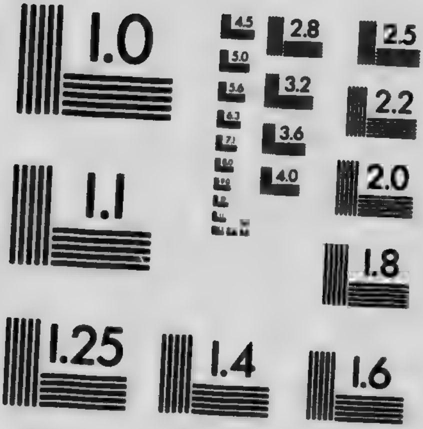
This may be permitted until the pupil can himself suggest, or is led to see, how space and labour may be saved by using the following form:

$$\begin{array}{rcrll} 496 & & & & \\ \underline{24} & & & & \\ 1984 & = & 4 & \text{times} & 496 \\ \underline{9920} & = & 20 & \text{"} & 496 \\ 11904 & = & 24 & \text{"} & 496. \end{array}$$

This is later abbreviated by leaving out the explanatory statements (4 times 496, etc.) and also by omitting the

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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zero in the second partial product since, as already learned, the zero is used only to keep the other digits in their proper places, and these places are marked by the digits in the first partial product.

The pupil can now multiply by any two-digit numbers, but *at first* the teacher will do well to give problems where the multipliers are selected with some definite purpose in view. For instance, in multipliers like 32, 54, 43, etc., there is an association which will be found helpful to the beginner.

THREE-DIGIT MULTIPLIERS

Three-digit multipliers can now be handled with ease. Multiply 743 by 876. Set down the work thus:

$$\begin{array}{r c r l l l}
743 & & & & & \\
876 & & & & & \\
\cline{1-1}
4458 & = & 6 & \text{times} & \text{the} & \text{multiplicand} \\
52010 & = & 70 & \text{``} & \text{``} & \text{``} \\
594400 & = & 800 & \text{``} & \text{``} & \text{``} \\
\cline{1-1} \cline{3-3}
650868 & = & 876 & \text{``} & \text{``} & \text{``}
\end{array}$$

The explanations are then omitted, as are also the unnecessary zeros in the second and third partial products.

The pupil may here be given the names applied to the lines of addends—*partial products.*

The teacher should be careful to place, and to see that pupils always place, the units' digit of the multiplier under the units' digit of the multiplicand, and the units' digit of the product under the units' digit of the multiplier and the multiplicand.

The work of multiplying, say 743 by 407, presents nothing new, nothing different from what has already been discussed.

However, to go into detail, it may be pointed out:

1. That the second partial product is 400 times 743;
2. That it is obtained in the manner indicated previously;
3. That (400 times 3) = (3×4×100) = 12 hundreds = 1 thousand and 2 hundreds;
4. That the 2 is placed under the hundreds of the first partial product;
5. That 4 times 4 gives 16; and that this is of the order next higher than hundreds, and so on.

$$\begin{array}{rrrrrr} & & & 7 & 4 & 3 \\ & & & 4 & 0 & 7 \\ \hline & & 5 & 2 & 0 & 1 \\ 2 & 9 & 7 & 2 & & \\ \hline 3 & 0 & 2 & 4 & 0 & 1 \end{array}$$

CHECKING MULTIPLICATION

1. Multiply the multiplier by the multiplicand, and compare the result with the one first obtained.
2. Employ the addition principle underlying the multiplication by any multiplier of two or more digits.
3. Multiply, beginning at the left of the multiplicand instead of at the right.

The third check may be used to explain the process of division.

ROMAN NOTATION

By calling the attention of the pupils to the dial of a clock, they will see that the numbers on it are not written in figures but in letters. A simple talk may then be given on the two kinds of Notation, *Arabic,* and *Roman.* On the clock dial the pupils will discover three different letters, I, V, and X, and will be prepared to learn that each of these is a symbol for a number, and that there are perhaps other letters, besides these, used as number symbols.

The numbers may then be taken up in their order:

I and its repetition; V and its combinations with I; X and its combinations with I; X and its combinations with V; X and its combinations with V and I; X and its repetitions.

The symbols will be introduced as they are required for the part of the number series with which the teacher intends to deal.

While the learning of Roman notation is, no doubt, mainly accomplished by a definite effort of memory, yet there are, underlying the system, some rational principles which the pupils may in time be led to understand, and which will enable them to contrast it with the ordinary Arabic system of notation.

1. The symbols may be placed in two classes:

(*a*) I, X, C, M; and

(*b*) V, L, D.

It will be seen that the symbols in class (*a*) correspond to our units, tens, hundreds, and thousands. The value of each symbol in this class is ten times that of the preceding symbol; the value of each symbol in class (*b*) is five times that of the corresponding symbol in class (*a*).

2. The symbols in class (*a*) are repeated, but those of class (*b*) are not.

3. Repeating a letter repeats its value. Thus, II represents 2.

4. A letter should not occur more than three times consecutively in the Roman numeral.

5. When a symbol representing a number of less value is placed *after* one representing a number of greater value, the number indicated is the *sum* of the value of such letters. Thus, CX represents 110.

6. When a letter of less value is placed *before* one of greater value, the number expressed is the *difference* between the value of such letters. Thus, XC represents 90.

7. One place in Arabic notation must always be represented by its Roman equivalent before [illegible]ing to the next place:

45=40+5=XLV.

CHAPTER X

DIVISION

ITS NATURE: TERMS

THERE are two problems in division, namely:

1. To find the number of groups into which a given number of objects may be divided if a stated number is put into each group.

2. To find the number of objects in each group if the given number of objects is divided into a given number of equal groups.

Examine the following problems:

1. How many pencils at 5 cents can be purchased with 40 cents?

2. If 40 apples be divided equally among 5 boys, how many will each have?

While these are different problems, their solutions, so far as the pupil is concerned, are essentially the same—the first being represented by $40 \div 5 = 8$, and the second by $\frac{1}{5}$ of $40 = 8$; that is both are solved by finding the co-factor of 5 with respect to 40.

The process of finding this co-factor is called *division* and its relation to multiplication is at once apparent. The product and one factor are given and the co-factor is to be determined. The product becomes the *dividend,* the given factor the *divisor,* and the required co-factor the *quotient.*

Sometimes the divisor is not contained an exact number of times in the dividend, and the problem becomes one of finding the multiple of the divisor nearest to and less than the dividend, and then the co-factor of that.

SHORT DIVISION

There does not seem to be any good reason for beginning formal division by the "long division" process; the notion that it is the easier process is largely imaginary. No serious difficulties are encountered in teaching "short division", and such little difficulties as may be encountered cannot be avoided by resorting to the long method. Oral naturally precedes written work in any topic on arithmetic, and it is evident that the necessity for the "long" process in division arises out of the occurrence of remainders and products too large to be carried in the mind.

Short division is the repetition of a single step, namely, the division of a number not greater than 119 by a number not greater than 12. So that, if the pupil has been taught to divide any number up to 119 by any number up to 12, he is able to divide any number whatever by any of the first twelve numbers.

Suppose 4987 is to be divided by 8, the work consists of the three steps: 49 divided by 8, 18 divided by 8, and 27 divided by 8.

Hence the first step in teaching division is to teach problems in which any of the first twelve numbers is a divisor, and the dividend a number such that the quotient will not exceed 9—that is, to divide any number up to 119 by [illegible], any number up to 109 by 11, any number up to 99 by 10, any number up to 89 by 9, etc.

If a topic in arithmetic is thoroughly taught, viewed from all sides, discussed in all its aspects, and considered in its possible connections with subsequent topics, the teaching of these subsequent topics will be made easier.

Thus, if children have been taught the multiplication tables thoroughly so that the mention of any two factors at once suggests the product, it will be an easy matter for them to recall the other factor when one factor and the product is given. Indeed, this should be incidentally learned with the multiplication tables.

INTRODUCTORY ORAL WORK

As a preparation for division the multiplication tables must, therefore, be thoroughly reviewed, and special attention given to such oral exercises as:

$6 \times ? = 24$.
How many 6's in 24?
What number taken 6 times will give 24?
Divide 24 into 6 groups.
Divide 24 by 6.

All of these lead to the usual form: $24 \div 6$.

A review of notation will also be valuable, in order to emphasize the facts illustrated by such questions as: How many tens in 200? Divide 20 tens by 5. How many tens in 294? Divide 29 tens by 6. How many ones in 5 tens. In 5 tens and 4? Divide 54 by 6, etc.

Formal division might then begin with the consideration of oral problems such as the following:

1. How many pencils at 5 cents each can be bought for 15 cents?

2. How many sheep at 8 dollars each can be purchased for 60 dollars?

3. Divide 50 apples among 6 girls without cutting any of the apples, giving each girl the same number and the greatest number possible.

4. How many teams of horses worth 8 hundred dollars a team can be bought for 32 hundred dollars?

5. Then how many teams of horses worth $800 a team can be bought for $3,200?

6. If it cost 24 million dollars to build 4 street railways, what would it cost to build one street railway?

7. Then if it cost $24,000,000 to build 4 street railways, what would it cost to build one street railway?

The pupil soon learns that 4×6 equals 24 and that 24 divided by 4 gives 6 as a quotient *whatever be the unit of measure used;* 24 thousands divided by 4 gives 6 thousand; 24 millions divided by 4 gives 6 millions; 49 hundreds divided by 8 gives 6 hundreds for quotient and 1 hundred remains.

Illustration

Divide 4984 by 8.

To make clear the operation, first multiply 623 by 8 beginning at the left of the multiplicand. Thus:

$$623\times8=(600+20+3)\times8=4800+160+24=4984.$$

If now 4984 is divided by 8, the process is reversed, thus:

$$4984\div8=(4800+160+24)\div8=600+20+3=623.$$

Here 48 hundreds, 16 tens, and 24 units are divided by 8, and the quotients 6 hundreds, 2 tens, and 3 units added.

The operation, therefore, contains three exercises, 48÷8, 16÷8, and 24÷8.

After a few like examples in which multiplication and division are correlated, the work may be set down in the usual form and explained thus:

Problem: To divide 4984 articles into 8 equal groups;

$$\begin{array}{r} 8\underline{)4984} \\ 623 \end{array}$$

The number of thousands, 4, is not sufficient to pe of putting a thousand in each group; changing the the next lower order we have with the 9 of that order 49, indeed the dividend is very commonly read as forty-nine hundred and eighty-four. To divide 49 hundreds by 8 is no more difficult than to divide 49 units by 8; the one hundred which remains is changed to tens, making with the 8 tens, 18 tens, and the next problem is to divide 18 tens by 8, which gives 2 as quotient and 2 tens for remainder; similarly, the 2 tens are changed to units, and with the 4 of that order give 24 units, which divided by 8 gives 3, the last figure in the quotient. No matter what the series of figures, the process is the same, and the pupil should experience no real difficulty if rational methods and practice have been followed.

SEAT WORK

The teacher should drill pupils very thoroughly in class on the work of short division before assigning them problems for seat work. Inexperienced teachers are apt to set pupils at the solution of problems before the process has been well taught. The *testing* phase of instruction is as important as the *teaching* phase, but the former must not be resorted to before the latter has been completed.

DIVISION BY 10, 20, ETC.

From a review of numeration and of multiplication by 10, 20, etc., the pupil should find it easy to divide by these divisors, using short division.

In dividing by 10 the quotient is the number of tens in the dividend, and this can be ascertained from numeration without division. Thus in 3674 there are 367 tens and 4, and so the quotient in dividing by 10 would be 367. If now the number was to be divided by 20 or 2 tens, the 367 tens must be placed in groups of 2 tens, that is, the 367 has to be divided by 2. There would be a remainder of 1 ten, which with the original 4 units makes a total remainder of 14 units.

LONG DIVISION

Connect long division with short division. One of the weak features of teaching arithmetic is to teach each topic as though the pupil had never learned any arithmetic before; little or no use is made of his previous knowledge; no effort is made to show how one topic is related to or dependent upon one previously taught.

The only difference between short and long division is to be found in the fact that whereas in short division the results of the necessary multiplications and subtractions are carried in mind, in long division these results are written down. This may be demonstrated by using the *long* form to work some of the short division exercises, thus:

```
8)4372                    546
  ----                 8)4372
   546  4                 40
                          --
                           37
                           32
                           --
                            52
                            48
                            --
                             4
```

Illustration

Divide 9878 by 31

Suppose the problem to be to divide 9878 articles among 31 persons. Correlate the solution with short division. Divide 98 hundred into 31 equal groups; how many can be put in each group? 3 hundreds. Where can the 3 be placed so as to show its denomination, or name? How many hundreds are left? 5 hundreds. Now, as in short division, what is done with the remainder, 5 hundreds. The 5 hundreds with the 7 tens in the dividend make how many tens? Divide 57 tens by 31; what is the quotient, and what is the remainder?

$$\begin{array}{r} 318 \\ 31\overline{)9878} \\ \underline{93} \\ 57 \\ \underline{31} \\ 268 \\ \underline{248} \\ 20 \end{array}$$

What is done with the remainder? The 26 tens with the 8 units in the dividend make how many units? Divide 268 units by 31; what is the quotient and what is the remainder?

DIFFICULTIES

After the pupil knows the first steps of long division, there still remain two difficulties to be overcome:

1. To tell the quotient figure;
2. To supply a zero in the quotient when the divisor is not contained in the part of the dividend to be divided.

TRIAL DIVISOR

1. The first difficulty is overcome by having the pupil make frequent *trials*. This is, after all, the only method

which is used. It is not uncommon for one skilled in division to place down a quotient figure and, after trial, find that it has to be changed. Practice, here as elsewhere, leads to perfection, and the pupil may be led, through practice, to see that he can approximate the quotient figure by using a *trial* divisor. For instance, in dividing by 30, 40, 50, etc., it has been shown that the quotient can be found by short division, as the divisors really used are 3, 4, 5, etc.

Now when the divisor is, say, 36, it is evident that the quotient will be less than if the divisor was 30 and greater than if the divisor was . Hence an *approximate* quotient would be found by dividing by 30 or by 40, that is, by 3 or by 4. The 3 or the 4 is then called the trial divisor.

Illustration: Placing a Zero in the Quotient

2. The second difficulty is overcome by making clear the place value of the digits in the partial dividend, for example: Divide 8569 by 21.

```
    408
21)8569
   84
   ---
    169
    168
    ---
      1
```

The first step is to divide 85 *hundreds* by 21; the quotient is 4 *hundreds* and the remainder 1 hundred; reducing this remainder to the next lower order we have 16 *tens*; this is not sufficiently large to contain 21, so we must reduce it to the next lower order; with the 9 units in the dividend, this gives us 169 *units*; 169 units divided by 21 give 8 and a remainder of 1. The 4 in the quotient is *hundreds*, and the 8, *units*. Emphasize these two facts rather than that there are *no tens*. The zero is written in the quotient to indicate more clearly the place value of the 4 and the 8.

SUGGESTIONS FOR LONG DIVISION

1. Proceed slowly at first.

2. Teach the pupil the formal steps of long division before letting him encounter either of the difficulties mentioned above.

3. The first examples in long division should have as divisors such numbers as 21, 31, 41, 51, and later, 42, 52, 62; with these divisors it is not difficult to determine the quotient figure—that is, to tell "how often it will go".

4. In the first examples in long division, zero should not occur in the quotient.

5. When the pupil has thoroughly learned the form of long division, he may be introduced to the difficulties.

6. He should be led to see that a good plan for keeping check on the number of figures in the quotient, as well as for keeping track of their place values, is to follow the rule of writing the first figure in the quotient above the one in the dividend that has the same place value, as was done in the examples above.

7. He should be frequently asked to interpret his answer. For example: $75 \div 5 = 15$ means that $75 = 15$ fives, or 5 fifteens.

CHAPTER XI

FACTORING AND CANCELLATION

FACTORS

A KNOWLEDGE of the multiplication tables naturally leads to the idea of *factor* and the introduction of the term. Thus, when it is known that 3×4=12, it should be explained that 3 and 4 are called *factors* of 12.

From division there is obtained a new view of a factor—that it is the same as an *exact divisor.*

From either point of view it will be seen that factors are found by *trial* multiplication or by *trial* division.

In making the trial the pupil is aided by his knowledge of the multiplication tables and by practice in multiplication and division. He soon learns to determine such factors as 2, 5, 10, but most factors are discovered *only by trial.*

PRIME FACTORS

From multiplication it will be seen that, when several numbers are to be multiplied together, the operation consists in multiplying two of the numbers together, then multiplying the product by the third number, and this product by the fourth number, and so on.

For example: $8\times7\times9\times5\times4=56\times9\times5\times4$

$$=504\times5\times4$$
$$=2520\times4$$
$$=10080.$$

To get the factors of a number the process is reversed, and there are obtained two factors, then three, then four, etc.

If then the *prime* factors of a number are required, the number is broken up into two factors one of which is *prime*, then into three factors two of which are prime, and so on until no more factors can be found.

Illustration

Find the prime factors of 1260.

By trial,
$$\begin{aligned} 1260 &= 2\times 630 \\ &= 2\times 2\times 315 \\ &= 2\times 2\times 3\times 105 \\ &= 2\times 2\times 3\times 3\times 35 \\ &= 2\times 2\times 3\times 3\times 5\times 7. \end{aligned}$$

This is usually set down in the following form:

$$\begin{array}{rr} 2) & \underline{1260} \\ 2) & \underline{630} \\ 3) & \underline{315} \\ 3) & \underline{105} \\ 5) & \underline{35} \\ & 7 \end{array}$$

Note that the prime factors of 1260 are 2, 2, 3, 3, 5, and 7, *not* 2 times 2 times 3 times 3 times 5 times 7.

PRIME AND COMPOSITE NUMBERS

Here the information should be given that a number that is exactly divisible only by itself and by 1 is called a *prime* number. It is usually said to have no factors. All other numbers are called *composite.*

Give examples of each class and have pupils do likewise.

Odd and even numbers should also be explained and examples given.

DIVISION BY FACTORS

Illustration

Example: Divide 5795 by 42, using factors.

6)5795
 7)965 groups of six with remainder of 5 ones,
 137 groups of forty-two with remainder of 6 sixes.

The whole remainder is 6 groups of six and 5 ones, or 41 ones—that is, the true remainder is 41.

Example: Divide 73205 by 168, using factors.

7)73205
6)10457 groups of seven with a remainder of 6 ones,
 4)1742 groups of forty-two with a remainder of 5
 435 groups of one hundred and sixty- [sevens,
 eight with remainder of 2 forty-twos.

The quotient is 435 and the remainder is: 2 forty-twos + 5 sevens + 6 ones, or 84 ones + 35 ones + 6 ones, or 125 ones, the true remainder, as it is usually called.

Do not teach the pupils a formal rule for finding the true remainder; show them that, in the example just given, ones have been put into groups of seven; the sevens combined to make greater groups, namely, groups of 6 sevens or forty-two; these, in turn, into groups of one hundred and sixty-eight, so that we have:

73205=435 groups of one hundred and sixty-eight
+2 groups of forty-two+5 sevens+6 ones,
=435 groups of one hundred and sixty-eight+125 ones.

Finding the true remainder is simply collecting the broken groups and counting them by ones.

ADVANTAGES

The advantages of teaching division by factors are:

1. It facilitates in many cases the operation of division.

2. It teaches the relationship of numbers, enables the pupil to understand the counting of objects by groups, makes him see the meaning of a scale of notation, and prepares him for reduction of denominate numbers from a lower to a higher unit.

CANCELLATION

Cancellation is the application to division of two principles:

1. If the dividend and divisor are divided by the same number the quotient is unchanged.

2. If the product of two or more numbers is to be divided by a given number the result may be obtained by dividing one of the numbers by the given number, and then multiply.

To teach the topic, therefore, all that is needed is to establish the principles and show how they may be applied to simplify division by substituting small numbers for larger ones.

Illustration

1. Divide 84 by 28. Divide each of these numbers by 7, and again divide. Compare this second quotient with the first.

Divide 96 by 16. Divide each of these numbers by 4, and again divide. Compare this second quotient with the first. Follow these by like examples.

Then ask if 36 is to be divided by 18 what other numbers may be found as above to give the same quotient, etc.

2. Divide (12×9) by 4. Now divide the 12 by 4 and multiply the quotient by 9. Compare results.

Divide (15×14) by 7. Now divide 14 by 7 and multiply the quotient by 15. Compare results. Follow by like exercises.

Now, in what two ways may (32×13) be divided by 8?

3. If then (35×24) is to be divided by (28×15), the operation may be simplified by dividing dividend and divisor by the same number, and this may be done by dividing any factor of the dividend and any factor of the divisor by that number.

By what number can a factor of the dividend and a factor of the divisor be divided? 3 or 4, or 5 or 7. Take 7 and the result will be (5×24)÷(4×15). This may be simplified by dividing dividend and divisor by 5, and the result will be (1×24)÷(4×3). This can be further simplified.

The work is usually set down thus:

$$\frac{\overset{\overset{1}{\cancel{5}}}{\cancel{35}} \times \overset{\overset{2}{\cancel{6}}}{\cancel{24}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{28}} \times \underset{\underset{1}{\cancel{3}}}{\cancel{15}}} = \frac{1 \times 2}{1 \times 1} = 2.$$

Follow by like examples and extend the operation to include numbers where the factors are not so apparent.

The quotient 1 should at first be written, else there will be acquired the idea that zero is the quotient when 5 is divided by 5, etc. The 1 may be dropped later.

CHAPTER XII

PROBLEMS

SOLUTIONS OF PROBLEMS

ALL arithmetical operations are studied and mastered chiefly for the purpose of applying them to the solution of problems and, consequently, throughout the whole course, carefully selected problems must be provided for such applications. It must be noted, however, *that the solution itself is of much more importance than the writing of it either on the black-board or on paper.* It is a dangerous practice to insist, from the very first, upon type forms for the statement of solutions. *The pupil must not be led to emphasize the form at the risk of neglecting the reasoning.*

It would seem much the better method to allow the pupil latitude in solving problems, and to accept any correct solution. This will teach him self-reliance and independence. The solution, at first, should be stated in the pupil's own words and style. If the statement can be improved, let him endeavour to re-word it for himself and again re-word it. In time he will express himself more clearly and concisely. He can be shown how his statement can be further improved in neatness and conciseness and be led to see the value of the type form, not as a means for getting the solution but as a means for expressing it.

It is evident that undue prominence given to any particular type of solution may result in discouraging originality and independent reasoning. On the other hand, it is desirable that the pupil be taught to give expression to his argument, either orally or in writing, in clear and concise terms; and for this purpose typical solutions may be useful for guidance.

But it is a mistake to think because the pupil can use the "type form" that he must of necessity have reasoned out the solution logically. As a matter of fact his reasoning may not have been directed to the solution at all, but may have been wholly concerned about *remembering* where, in the type, the different figures should go and where multiplication, division, or other operations were demanded.

CRITICISM OF TYPE SOLUTIONS

For an example take the problem: Find the cost of 6 oranges if 4 oranges cost 20 cents.

The work is invariably set down in some such form as this:

A. If 4 oranges cost 20c.,
1 orange will cost 20c.÷4, or 5c.,
and 6 oranges will cost 5c.,×6, or 30c.

Here it is an easy matter to *memorize* the *form,* put the figures in the correct places, and remember that the first operation was to divide and the second to multiply.

But the important question is: Has the pupil reasoned? The answer is obtained by an *oral* rather than by a *written* examination.

If left to himself, the pupil will often surprise the teacher with a variety of ways in which a solution may be clearly expressed. Note the following solutions for the above simple problem:

B. Since there are 4 oranges,
The cost is divided equally into 4 parts,
Thus placing in each part 5c.
When there are 6 oranges,
The cost will contain 6 parts,
Which will require 5c×6, or 30c.

C. 6 oranges=4 oranges + $\frac{1}{2}$ of 4 oranges,
Hence, cost=20c+$\frac{1}{2}$ of 20c=30c.

D. 1 orange =$\frac{1}{4}$ of 4 oranges,
∴ cost of 1 orange =$\frac{1}{4}$ of 20c, or 5c;
6 oranges =6 times 1 orange,
∴ cost of 6 oranges =6 times 5c, or 30c.

E. 6 oranges =$\frac{6}{4}$ of 4 oranges,
∴ cost of 6 oranges =$\frac{6}{4}$ of 20c, or 30c.

UNITARY METHOD

It is evident that C and D require a knowledge of fractions. But the point intended to be emphasized is that *fixed* types may mislead if indeed they are not dangerous.

The types illustrated above in A and D are frequently referred to as types of the "Unitary Method", a method in which the first operation is to find an expression for the unit by which a *given*, or known, quantity is measured in terms of the units by which the *required*, or unknown quantity is measured. The unit is generally *one*; but it should not be forgotten that any number may be taken as a unit of measurement, and that C and E are equally good types of a unitary method, as easily understood as A and D, and have the advantage of brevity.

ORAL, OR MENTAL, ARITHMETIC

Too much stress cannot be laid upon the importance of oral, or mental, arithmetic. Indeed it will be found generally true that, in schools where arithmetic is weak, oral arithmetic has been entirely neglected or given no systematic attention. For the pupils, oral arithmetic is more interesting, stimulating and, on the whole, more useful than written arithmetic.

The pupil should be led to outline orally his solution of problems without reference to the numbers employed, and when he can do this satisfactorily, he need be required to write only that stage in which the numbers are represented, the operations indicated, and the final result obtained.

HOW TO ATTACK A PROBLEM

To attack a problem successfully a pupil must know:

1. What is the quantity to be determined;
2. What are the specific conditions given in the problem to enable him to determine the quantity;
3. What other conditions or relations are given, expressed, implied, or to be determined, connecting these specific conditions with the result to be obtained.

The pupil should be taught to look for and point out these three essentials in every problem set before him. For example, take the simple problem given above. The quantity to be determined is the sum of money which will buy 6 oranges. The condition given is that 4 oranges cost 20c. The relation in this case to be determined—that which connects this condition with the result to be found—is the relation between 4 oranges and 6 oranges.

STEPS IN A SOLUTION

The complete solution of a problem involves the following steps:

1. Understanding the problem,
2. Planning the work,
3. Executing the plan,
4. Testing the result.

Pupils should be given practice in each of the above steps. In solving problems he should be asked sometimes to write out a *full analysis* of his work, sometimes to give an *oral explanation* of how the problem is solved, and at other times to do only the *figuring* required to get the result. Each of these exercises is of equal importance. The business man contents himself with the last named.

DIFFICULTIES

These steps suggest the source to which may be attributed the chief difficulties experienced by pupils. These difficulties are:

1. Failing to understand the terms in which the problem is expressed, and the nature of the transaction involved;

2. Failing to plan the work owing to inability to discover the given condition, the quantity to be determined, and the connecting relations, and thus failing to decide upon the operation involved;

3. Failing to execute the plan owing to lack of proficiency in the mechanical work of addition, subtraction, multiplication, etc.;

4. Failing to apply proper checks to the result in order to determine its accuracy and reasonableness.

It is the duty of the teacher to endeavour to trace all difficulties to their proper source, to aid the pupil to remove them, and then to leave him to work his own way to the end of the solution without further assistance.

IMPORTANT POINTS IN A SOLUTION

In estimating the work of the pupils in the solution of problems emphasis should be placed upon the following points:

1. The knowledge of underlying principles;
2. The logical expression of the argument, involving a clear conception (*a*) of the result to be obtained, (*b*) of the procedure necessary to attain that result, and (*c*) of the logical order and arrangement of that procedure;
3. The accuracy of the answer;
4. The rapidity with which the answer was obtained, including a knowledge of oral and written methods upon which brevity depends;
5. Neatness of written work, including good penmanship;
6. Clearness and accuracy of statement, either oral or written.

COMPLEX PROBLEMS SIMPLIFIED

The solution of any problem, no matter how complicated, involves nothing but the solution of a series of simple problems each requiring but one step.

This being the case, a useful and instructive exercise on such complicated problems would be to have the pupils form the series of simple problems involved.

Illustration

For example, take the following:

For a bargain sale a furniture dealer reduced the price of tables from $28 to $21. Regularly, his sales were 3 tables a day, but during the bargain sale they were 12 a day. If the cost of the tables was $16.80 each, by what per cent. were his daily profits increased or decreased?

The simple problems are:

1. What was the original profit on a table?
2. What was the daily profit?
3. What was the reduced profit on a table?
4. What was the daily profit?
5. What was the difference between the profits?
6. What fraction is this difference of the first daily profit?
7. What per cent.?

THE SELECTION OF PROBLEMS

The teacher who would create an interest in arithmetic must not depend upon the text-book for suitable problems. The text-book must contain a more or less general selection, and cannot, therefore, adapt itself to the needs of the particular communities.

Problems should be taken from those activities which are most in evidence in the neighbourhood of the school, and the material for these should be collected or furnished by the pupils themselves. For this purpose the teacher would do well to keep a special note-book for local arithmetic. One problem pertaining to the life experiences of the pupil will be found more valuable than a dozen found only in text-books.

"SHARING" PROBLEMS

"Sharing" problems as arithmetical exercises have for a long period been popular with teachers and in text-books. A goodly number of them, at least in their customary form, are strictly *school* problems and find no place in the actual experiences of child or adult, nor are they of sufficient importance, considering the difficulties they afford or the arithmetical principles they involve, to warrant the undue prominence given them.

They are dealt with here to show their simplicity and, in some cases, to show the use which can be made of graphic illustrations—aids of which teacher and pupil should frequently avail themselves in the solution of problems.

THE THREE CLASSES

"Sharing" problems may be divided into three classes:

1. Those in which one share exceeds another by a stated sum;
2. Those in which the parts into which the whole is to be divided bear a given ratio to one another;
3. Those in which the first two conditions are combined.

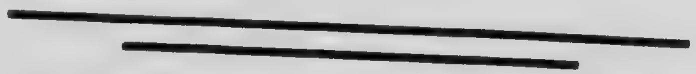
Examples:

1. Two pieces of cloth together contain 50 yards; the first is ten yards longer than the other; find the length of each.

2. Three bales of hay together weigh 335 lb.; the first weighs 40 lb. more than the third; the second weighs 25 lb. more than the third; find the weight of each.
3. A man, a youth, and a boy together weigh 390 lb. The youth weighs 55 lb. more than the boy and 70 lb. less than the man; find the weight of each.
4. Divide 120 marbles between two boys, giving one 5 marbles as often as the other is given 7.
5. Divide $60 between two persons, giving one three times as much as the other.
6. Divide $240 among three persons in the proportion 4, 5, 6.
7. Divide $62 between two persons, giving one $10 more than three times what the other is given.
8. Three houses are together worth $16,410. The first is worth twice as much as the other two together, and the second is worth $570 more than the third; find the value of each.

SOLUTIONS

The first three problems are typical of the first class; 3 is slightly more complicated than either of the other two, but it is easily reducible to the same form as 1 and 2. The solutions may be graphically illustrated and taught in the following manner:



1. Let two lines represent the lengths of the pieces of cloth.

What is the total length of the two pieces? How much must be cut off the longer so that the remainder may be of the same length as the shorter?

What then is the total length of the two pieces?

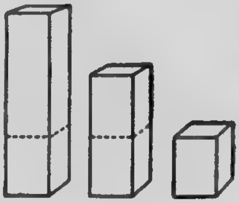
What is then the length of one of them?

What then was the length of the other before it was shortened?

50 yards = length of the two pieces.

40 yards = length of the two pieces each the same length as the shorter one.

Therefore 20 yards is the length of the shorter piece, and 30 yards is the length of the longer piece.



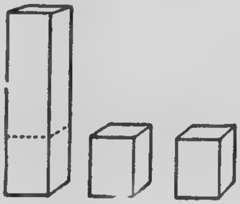
2. Represent the three bales graphically.

What is the total weight of the three?

What amount must be cut off the second so that the remaining portion may be of the same weight as the smallest bale?

What then will be the total weight of the three?

What then will be their respective sizes?



What amount must be taken off the largest bale so that the remainder may be of the same weight as the smallest?

What then will be the total weight of the three?

Compare their sizes now.

335 lb. = the weight of the three original bales.

25 lb. = the amount cut off the middle bale.

310 lb. = the weight of one large bale and two small ones.

40 lb. = the amount cut off the first bale.

270 lb. = the weight of three small bales.

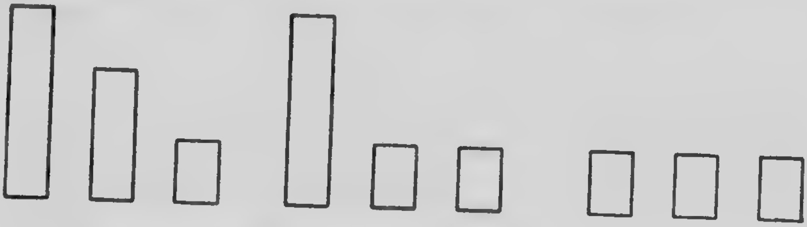
Therefore 90 lb. is the weight of the small bale.

Have pupils find the weights of the other two, and teach them the way to test the answers for accuracy.

3. What is the weight of the three, the man, the youth, and the boy? If we take the youth off the scales and put the boy in his place, by how much has the total weight been reduced?

What do the scales now register?

If we now take the man off the scales and put the boy in his place, by how much has the total weight been reduced?



What do the scales now register?

What does this weight represent?

What weight can then be found?

What weight shall we find next? and next?

$$\begin{array}{r} 390 \\ \underline{55} \\ 335 \\ \underline{125} \\ 210 \\ 70 \\ 125 \\ 195 \end{array}$$

The numbers to the left occur in the written solution. Have the pupils interpret each one. Have them state what each one tells and how it was obtained.

There is a feature in this problem somewhat different from that in the previous two. In this one we have to find how much the greatest share (the man's weight) exceeds the smallest share (the boy's weight); in the previous problems that amount was given directly.

It is not necessary, though it is always advisable, to make the smallest share the basis of the calculation.

Problems 4, 5, and 6 are examples of the third class; the method of stating ratio differs.

Problem 4 may be solved by supposing the marbles to be distributed between the two boys, giving 5 to one and 7 to the other, in turn, until all have been distributed; then, counting the number given to each.

After 5 marbles have been given to the first, and 7 to the second, there are 108 marbles left; after 5 have been given a second time to the first, and 7 to the other, there are 96 left; and the question of telling how many distributions of this kind we may make presents itself naturally.

Twelve marbles are given to the two boys in a distribution, and we therefore can make 10 distributions; since the first boy gets 5 marbles at each distribution, he receives altogether 5 marbles ten times, or 50 in all, etc.

Problem 5 means that the first person gets $1 as often as the second gets $3; in problem 6 one person gets $4 as often as the second gets $5 and the third $6; the remaining parts of the solutions resemble that of problem 4. There is a special solution of problem 5 a little simpler than this one, but it is doubtful if it be advisable to teach it at this stage.

Problem 7 may be solved by calling the small share a unit; then the larger share is 3 units and $10. The total amount of money, if measured in units, contains 4 units and $10. If measured in dollars, 62 is the measure of it—that is, 4 units and $10 are together equal to $62; $62 is greater than 4 units by $10—that is, 4 units must equal $52, etc.

To lead the pupils to understand clearly what is meant by *unit* in this case, the teacher may have them measure the length of their slates, using the length of their pencils as the unit; or the capacity of a pail, using a tin can as the unit; and express the result of their measurement orally, thus: the length of my slate is 3 when I use the lead-pencil as the unit of measurement; the capacity of the pail is 23 if I use the drinking-cup as the unit.

Problem 8 may be solved in two parts—by getting the value of the most expensive house first, and, secondly, the value of each of the others. Let us suppose we are buying three houses; we pay $2 for the first as often as we pay $1 for the other two together. Let us divide the total sum into two piles—one the price of the most expensive house, and the other the price of the remaining houses. To distribute the money into two piles we put $2 into one and $1 into the other; $3 are distributed in every such distribution, and this can be done 5,470 times; the first house is, therefore, worth 5,470 times $2, or $10,940. The value of the second and third together is $5,470; the difference in their values is $570. Call the value of the third a unit. What is the value of the two together? Two such units and $570. What is the value of the houses, expressed in dollars? $5,470. Then 2 units and $570 must mean the same sum as $5,470, and two units must mean a sum less than $5,470 by $570; these two units must equal $4,900 and one of them equals $2,450; that is, the third house is worth $2,450, and the second $3,020.

Later, when the pupil acquires a knowledge of fractions, it should be shown, in problems such as number 4, that since the first boy receives 5 marbles at every distribution, and that 12 marbles is the number distributed each time, the first boy receives $\frac{5}{12}$ and the second $\frac{7}{12}$ of the number distributed. Similarly, in problem 6, since the first person gets $4 every time the second receives $5 and the third $6, he receives $\frac{4}{15}$ of each distribution and $\frac{4}{15}$ of the whole.

CHAPTER XIII

COMPOUND RULES

TABLES

THE necessity for standard units can be shown by measuring the water in a vessel by cups of various sizes, glasses, saucepans, etc. A different number expresses the measure of capacity in each case, and in order that the measure of the capacity of any vessel may be intelligible to every person, a standard unit that every one knows must be used. Every school should be provided with pint, quart, and gallon measures; scales capable of weighing ounces and pounds; a foot rule marked in inch lengths, a yard-stick marked in foot lengths, and a sixty-six foot tape-line.

The pupil must be shown the different units which are used for any table. He will then arrange and name these units in their order, from the smallest to the largest, or vice versa; then, by actual use of the units, he will discover for himself the relation between any one unit and that of the next higher or lower order, and will summarize his results in the form of a table which he must memorize.

The first steps in Reduction will involve this actual measuring and the abstract solution of such easy problems as:

How many pint bottles can be filled from the milk in two quart bottles?

How many quart bottles can be filled from the milk in a five-gallon can?

Which is the greater, two gallons or fifteen pints, and by how much?

Express 3 gal., 4 gal., 5 gal., 8 gal., in quarts.

What is the equivalent in quarts of 5 pints, 8 pints, 10 pints, 17 pints?

Express the quantity 4 gal. in three different simple denominate numbers.

Express the quantity 24 pints in two other ways.

Treat the first part of linear measure in a similar way.

These two tables are selected for beginners because of the familiarity of pupils with the units and because of the simple scales of each.

In the application of simple multiplication and division the teacher should make use of problems involving the reduction from one unit to the next higher or lower unit. If this is done intelligently, further progress in Reduction, Descending and Ascending, is made comparatively easy.

ADDITION

As an introduction, use simple addition and show how it may be converted into compound.

Illustration

$$\begin{array}{lrrrrr} \text{Add} & & 4 & 6 & 8 & 3 \\ & & 7 & 2 & 9 & 6 \\ & & 2 & 7 & 8 & 5 \\ & & 3 & 8 & 2 & 7 \\ \hline & 1 & 8 & 5 & 9 & 1 \end{array}$$

What is the name given to the first column? To the second, third, fourth?

What table can be formed to connect units, tens, hundreds, etc.?

The above addition question may then be written:

Th.	H.	T.	U.
4	6	8	3
7	2	9	6
2	7	8	5
3	8	2	7
16	23	27	21

What do you do with the 21 units? Why? What is done with the 2 tens? Why? How many tens are there altogether? What is done with the 29 tens? Why? Etc.

Give like examples and then introduce the following:

Add:	4 yd.	2 ft.	10 in.
	3 “	2 “	6 “
	7 “	1 “	11 “
	3 “	2 “	8 “
	17 “	7 “	35 “

17 yd. 7 ft. 35 in. expresses the sum of the four addends, but not in conventional form.

4 yd.	2 ft.	10 in.
3 “	2 “	6 “
7 “	1 “	11 “
3 “	2 “	8 “
20 “	0 “	11 “

The conventional form is obtained by changing the 35 inches to feet and then the 9 feet to yards. The inch column is added and the sum changed to feet and inches, just as in simple addition the units are changed to tens and units. As the number of units is put in the proper place and the number of tens “carried”, so here the number of inches is written in the inch column and the number of feet carried to the next column. Beginners occasionally make the error of saying that 35 inches equal 3 feet and 5 inches; they do not at once realize that the scale is 12, not 10.

SUBTRACTION

$$\begin{array}{lll} 11 \text{ yd.} & 1 \text{ ft.} & 6 \text{ in.} \\ 4 \text{ ``} & 2 \text{ ``} & 8 \text{ ``} \\ \hline 6 \text{ ``} & 1 \text{ ``} & 10 \text{ ``} \end{array}$$

Here, if we employ the Additive method of subtraction, our problem is to find the quantity which must be added to 4 yd. 2 ft. 8 in. to give the sum of 11 yd. 1 ft. 6 in.

Now, just as in simple subtraction when the units' digit in the minuend was less than the units' digit in the subtrahend, we added enough to the subtrahend digit to make 1 ten + the minuend digit, so here we add enough to 8 in. to make 1 ft. + 6 in., that is, we add 10 in.

Adding the 10 in. to the 8 in. we get 1 ft. 6 in., and find that 1 ft. has to be carried to the 2 ft. in the subtrahend, giving 3 ft.

Again, since 1 ft. is less than 3 ft., we add enough to 1 ft. to give 1 yd. + 1 ft., that is, we add 1 yd.

Adding 1 ft. to the 3 ft., we get 1 yd. 1 ft., and find that 1 yd. has to be carried to the 4 yd. in the subtrahend, making 5 yd. To the 5 yd. we add 6 yd. to make the 11 yd. Hence, our remainder is 6 yd. 1 ft. 10 in.

If, on the other hand, we use the Decomposition method, we cannot take 8 in. from 6 in., so we change the 1 ft. to 12 in., and these, with the 6 we have, make 18 in.; 8 in. from 18 in. leaves 10 in. There are now no feet in the minuend, so we change the 11 yd. to 10 yd. 3 ft.; 2 ft. from 3 ft. leaves 1 ft.; 4 yd. from 10 yd. leaves 6 yd. The similarity to simple subtraction should be kept in mind by the teacher and impressed upon the pupil.

REDUCTION

1. Reduce 1411 pt. to gallons.

pt. pt.
2)1411
qt. 4)705 times ∴ 705 qt. and 1 pt.
176 times ∴ 176 gal. and 1 qt.

	pt.		gal.	qt.	pt.
∴	1411	=	176	1	1.

There are as many quarts in 1411 pints as there are groups of two pints in 1411 pints. The first problem is to divide 1411 into groups of two (division); since there are 705 such groups in 1411 and 1 single pint besides, therefore 1411 pt. equals 705 qt. 1 pt.; the remainder of the solution consists of similar reasoning.

Always insist on the pupil making such a summary as that contained in the last line of the solution—namely, 1411 pt. = 176 gal. 1 qt. 1 pt.

2. (*a*) Reduce 7 yd. 2 ft. 7 in. to in.

7 yd. = 21 ft. = 21 times 12 in. = 252 in.
2 ft. = 2 times 12 in. = 24 in.
7 in. = 7 in.
∴ 7 yd. 2 ft. 7 in. = 283 in.

Or (*b*)

7 yd. = (3 ft. × 7) = 21 ft.
21 ft. + 2 ft. = 23 ft.
23 ft. = (12 in. × 23) = 276 in.
276 in. + 7 in. = 283 in.

The work of this problem is usually written in the following form:

```
7yd. 2 ft. 7 in.
 3
23 ft.
12
283 in.
```

7 yd.=(7 times 3) ft. Since 7 times 3=3 times 7, we make the 3 the multiplier in this case so that the position of the 7 need not be changed. 23 ft. = 7 yd. 2 ft. Now 23 ft. = (23 times 12) inches; since 23 times 12=12 times 23, we make 12 the multiplier so that the position of the 23 need not be changed. 23 ft. = 276 in. In addition to this, there are 7 inches, making a total of 283 inches. Always insist on the pupil making a summarizing statement at the conclusion of his solution—for example, 7 yd. 2 ft. 7 in. = 283 inches.

3. Reduce 84674 minutes to weeks. In problems such as this, where it is necessary to use a divisor of two digits, we may divide by factors or divide by the long form and set down the quotient and remainder.

```
       6 0 ) 8 4 6 7 4 min.
          (4 ) 1 4 1 1 hr. 14 min.
       24 {
          ( 6 ) 3 5 2     —   8 )
              7 ) 5 8 dy. —   4 } 19 hr.
                  8 wk. 2 dy.
   .'. 84674 min. = 8 wk. 2 dy. 19 hr. 14 min.
```

MULTIPLICATION

Multiply 14 lb. 10 oz. by 9.

```
14 lb. 10 oz.
           9
131 "   10 "
```

Nine times 10 oz. = 90 oz. = 5 lb. 10 oz. Nine times 14 lb. = 126 lb., to which "carry" the 5 lb., and we get 131 lb.

DIVISION

Divide 85 gal. into 23 equal parts.

We can divide 85 gal. by 23; the quotient is 3 gal. and there are 16 gal. left. We can divide further by reducing 16 gal. to quarts. The problem now becomes one of dividing 64 qt. bv 23; the quotient is 2 qt. and remainder 18 qt. We can divide further by reducing 18 qt. to 36 pt.; we stop after the 36 pt. have been divided, because there is no limit less than pints.

23)85 gal. (3 gal. 2 qt. 1 pt.
69
16 gal.
64 qt.
46 "
18 "
36 pt.
23 "
13 "

Review work by getting answers from pupils to the following questions relative to the numbers used in the solutions, namely:

What is the 69 gal.? What is the 46 qt.? Where was 64 qt. obtained, and what therefore is it? When was 18 qt. the remainder? What is the nearest quantity to 85 gal., and less than it, that can be divided exactly by 23?

REDUCTION

Reduction of yards to rods, and the converse.

1 rod =5½ yd.
2 rods=twice 5½yd.—that is, 11 yd.
3 rods=3 times 5½ yd.—that is, 16½ yd.

1. Reduce 2 miles 36 rods 5 yd. 2 ft. to feet.

2 mi.
320
640
36
676 rods
5½
338
3380
3718
5
3723 yd.
3
11169
2
11171 feet

Note (1): In this reduction we are to think of the operation as signifying 2 × 320 rods; or by the laws of commutation as 320 × 2 rods, and not as representing 320 × 2 miles.

2 miles = twice 320 rods = 640 rods.
676 rods = 676 × 5½ yd. = 676 half-yd. + 676 times 5 yd. = 3718 yd.

Note (2): The pupils should be taught later to add in the 36, the 5, and the 2 without writing them.

Note (3): Teach the pupils to establish this table, it will be convenient to know it: ½ yd. = 1½ ft. = 1 ft. 6 in. = 18 in.

2. Reduce 30 yd. to rods.

To reduce 30 yd. to rods it is necessary to divide 30 into groups, each containing 5½ yd. This is more easily done by changing 30 yd. to half-yd. and then dividing it into groups of 11 half-yd.

30 yd. = 60 half-yd. = 5 rods and 5 half-yd. or 5 rods and 2½ yd. = 5 rods. 2 yd. 1 ft. 6 in.

3. Reduce 242337 inches to miles, etc.

in.
12)242337 in.
ft.
3)20194 times, ∴ 20194 ft. and 9 in.
5½ yd.)6731 times, ∴ 6731 yd. 1 ft.
or 11 half-yd.)13462 half-yd.
320 rd.) 1223 times, ∴ 1223 rods and 9 half-yd. or 4½ yd.
3 times, ∴ 3 miles 263 rods.

242337 in. = 20194 ft. 9 in.
20194 ft. = 6731 yd. 1 ft.
6731 yd. = 1223 rd. 4½ yd.

1223 yd. = 3 mi. 263 rd.
242337 in. = 3 mi. 263 rd. $4\frac{1}{2}$ yd. 1 ft. 9 in.,
but $\frac{1}{2}$ yd. = 1 ft. 6 in.,
∴ 242337 in. = 3 mi. 263 rd. 5 yd. 3 in.

We reduce a quantity expressed in terms of a smaller unit to larger units in order to get a more definite idea of its value. Thus, we form no definite idea of a distance between two points when we are told that it is 242,337 inches; but we have a more definite idea of the same distance when we are told that it is three miles, 263 rods, 5 yards, 3 inches.

UNITS

A little practice in reduction ascending and descending will lead the pupil to appreciate the fact that all multiplication is reduction descending and all division reduction ascending. In multiplication we are changing from a large (derived) unit to a small (primary) unit, and in division from a small (primary) unit to a large (derived) unit. It will also be learned that there is no *fixed* unit. That sometimes it is 1 (in.), sometimes 12 (in.), sometimes 3 (ft.), sometimes 5280 (ft.). It will be seen, therefore, that any quantity may be taken as a unit, but that for some of these there is no name, such as foot, mile, pint, etc. Here, too, it may be made clear that the *measured quantity* does not change but the *measure* changes according to the unit used in measuring or, as it is usually called, the unit of measurement. For instance, the quantity is the same whether we say 3 (bu.) or 12 (pk.), but the measure 3 is less than the measure 12, because the unit of measurement, 1 bushel, is greater than the unit of measurement, 1 peck.

CHAPTER XIV

SQUARE MEASURE AND CUBIC MEASURE

A SOLID

THE first lesson on surface should be begun with a brief consideration of solids. Pupils are led to see that a solid is anything that occupies space. A chalk-box is held in front of the class, then removed to some other position, and pupils are led to see that the space occupied by it in the first instance is afterwards occupied by air. A book, a ball, a piece of chalk are then moved from position to position in a similar way to illustrate this point.

Pupils are shown that objects are of various shapes, and are told that the one under consideration on the particular occasion has the shape of the chalk-box. Have them name several objects in the school-room that are of this shape. A sheet of paper, the slate out of which the blackboard is made, are each an example of this shape. Point out that the sheet of paper has thickness, although its thickness is small compared with length and width. The volume of a solid is the amount of space occupied by it. When we speak of the volume of a solid we mean its " bigness ", not its length, or its width, or its thickness, but its size as a whole.

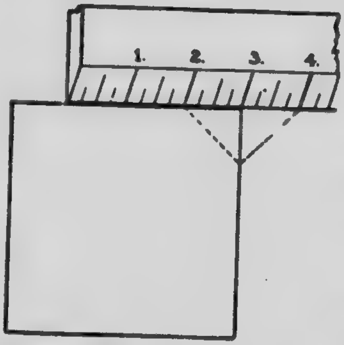
SURFACE

Show pupils that the surface of a solid usually spoken of as the " outside ", the " top ", etc., is the boundary which separates the solid from the space which it does not occupy; that the surface of the black-board is that which separates the slate from the air in the room; that the surface of the floor is the boundary between it and the air above it; that the surface of the ball is the boundary between it and the air around it; that some surfaces are flat like those of black-boards, and floors, and that the flat ones will be considered first.

Have pupils mark off a *portion* of the black-board surface by drawing, (1) curved lines, (2) straight lines, (3) four straight lines, and (4) four straight lines that meet in the same way as do the edges of a picture-frame. Show pupils that the boundaries between one portion of space and another are lines; that the school grounds are separated from the road or street by a line; that lines are boundaries between portions of space marked off on the black-board and the rest of its surface. Show them that the surface of the chalk-box is divided into six portions, and that each of these is bounded by four straight lines.

RIGHT ANGLES AND RECTANGLES

When the teacher is telling the pupils what is meant by a right angle he should keep strictly in mind the geometrical definition of it. It can be illustrated by showing the angle of a carpenter's square or by laying the edge of a rule along the edge of a piece of cardboard.



Tell the pupils that, if the corner of the card is of the same size as the corner between the ruler and the edge of the card, the corners are right angles. Illustrate further by placing the ruler similarly on the upper edge of the open door. Show them that, if a carpenter cuts a board in two, he can tell whether the corners are right angles or not by placing one piece on top of the other and noticing whether or not the corners exactly fit.

A portion of a flat surface that is inclosed by four straight lines and has its four corners right angles is called a *rectangle.* If the four lines are of equal length, the rectangle is a *square*; if only the opposite sides are equal, it is called an *oblong.* The length and width of a rectangle are called its *dimensions.*

AREA

When we wish to find out how heavy an object is, we speak of finding its *weight.* When we wish to find out how much it will hold, we speak of finding its *capacity.* So, too, when we wish to find out how much surface it has, we speak of finding its *area.*

The ordinary units of area are square inch, square foot, etc., just as ton, lb., oz., are the units of weight. A square inch is the amount of surface contained in a rectangle each of whose dimensions is one inch; a square foot is the amount of surface contained in a rectangle each of whose dimensions is one foot; a square yard is the amount of surface contained in a rectangle each of whose dimensions is one yard.

THE INCH SQUARE AND THE SQUARE INCH

The difference between an inch *square* and a square *inch* should be carefully pointed out. The inch square emphasizes the *shape* of the surface. It is always a *square* whose sides are each one inch in length.

A square inch emphasizes the *size* of the surface no matter what may be its shape. It is the *amount* of that surface which, if it were shaped into a square, would make exactly an inch square.

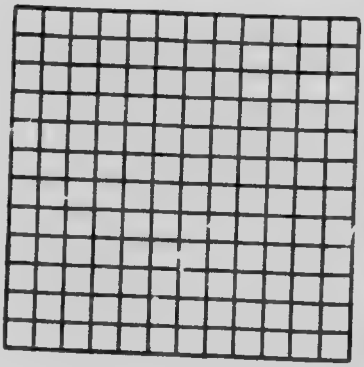
This difference is made apparent by drawing figures to represent (*a*) two square *inches* and (*b*) a two-inch *square.*

TABLE OF SQUARE MEASURE

SQUARE FOOT, SQUARE YARD

Draw, very accurately, three rectangles to represent, (*a*) the square inch, (*b*) the square foot, and (*c*) the square yard; have the pupils restate the dimensions of each and the area of each.

Divide the sides of the second rectangle into twelve equal parts, join the corresponding points, and thus divide the figure, (*a*) into strips, and (*b*) each strip into a number of little squares. Have the pupils count the number of squares, and lead them to see how they might have computed the number (12 rows of 12 each).

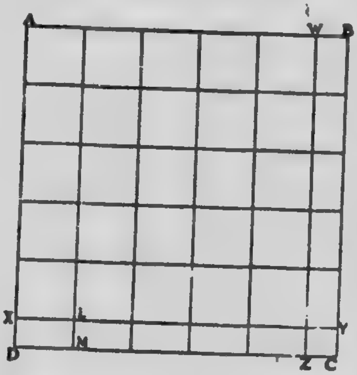


Since each of these little squares is a square inch in area, 144 sq. in. equal 1 sq. ft.

Have the pupils divide the square yard into square feet, and ascertain the number that is equivalent to one square yard.

SQUARE ROD

Draw a rectangle a rod wide and a rod long on the floor of the class-room or in the school yard. It may be assumed that the edges of the boards in the floor meet the walls of the room at right angles. Draw the diagram as accurately as possible, the pupils assisting in the work. Divide the sides of the rectangle into six parts, five of a yard in length, and one of half a yard in length; join the corresponding points of the sections as in the diagram below.



In the ordinary class-room, the floor space underneath the desks will be used for the diagram. The teacher should plan, if possible, to have the portion X Y C D fall on an open space at the front of the room and the portion W B C Z on an open floor space at the side of the room. A diagram on the floor showing the actual size of a square rod is preferable to a diagram drawn to a scale on a black-board. When the diagram is completed, compute its area in square yards. Have the pupils give dimensions of each of the thirty-six rectangles into which the whole has been divided. The area of each of the largest ones is one square yard. There are 25 of these; there are 10 of the size of L M D X. Lead pupils to see that two of these latter placed side by side would make a rectangle equal in area to one of the former, and that therefore one is half a square yard, and the total area of the ten is 5 sq. yd. The smallest square of all is the same width as the one lying beside it and half as long; lead the pupils to see that the area of the one is half that of the other and therefore one quarter of a square yard.

Finally, the pupils will see that 30¼ sq. yd. is equal to one square rod.

OTHER UNITS

Tell the pupils that we use another unit of area, namely an acre, for which there is no corresponding linear unit; tell them it is a portion of surface equal to 160 sq. rd. Give them a definite idea of this area by actually measuring it and by referring to neighbouring fields. The dimensions of these fields should be used in the class problems.

With the assistance of a diagram teach pupils to compute:

1. The number of sq. rd. in a sq. mi.
2. The number of acres in a sq. mi.

This completes the teaching of the table, and if it has been well done, the pupils have incidentally learned the first step in finding the area of rectangles.

REDUCTION

The reduction of square yards to rods and the converse presents little more difficulty than the reduction of linear yards to rods. Pupils should be taught to construct the following table:

1 sq. yard=9 sq. ft.
½ sq. yard=4½ sq. ft.=4 sq. ft. 72 sq. in.
¼ sq. yard=2¼ sq. ft.=2 sq. ft. 36 sq. in.
¾ sq. yard=6¾ sq. ft.=6 sq. ft. 108 sq. in.

Example: Reduce 688147 sq. in. to rods.

144) 688147 sq. in.
9) 4778 sq. ft. 115 sq. in.
530 sq. yd. 8 sq. ft
4
121) 2120 quarter-yd.
17 sq. rd. 63 quarter-yd. or $15\frac{3}{4}$ sq. yd.

$30\frac{1}{4}$
4
121 quarter-yd.

∴ 688147 sq. in. = 17 sq. rd. $15\frac{3}{4}$ sq. yd. 8 sq. ft. 115 sq. in.,

or 17 sq. rd. 15 sq. yd. 8 sq. ft. 115 sq. in. + 6 sq. ft. 108 sq. in.,

or 17 sq. rd. 16 sq. yd. 6 sq. ft. 79 sq. in.

The operation of dividing by 144 and 121 may be performed by the long method in another part of the work book and the result written in place as was done above. To divide by factors is to complicate a solution that is already somewhat involved.

CUBIC MEASURE

Apparatus. A cubical block, 1 ft. edge, and a number of cubes of 1 in. edge are required for properly teaching this table, as well as for teaching the estimated volume. The usefulness of the large block is increased if a "layer" 1 ft. × 1 ft. × 1 in. can be detached from it.

The *volume* of a solid is the amount of space it occupies.

The units used in measuring volume are, cubic inch, cubic foot, cubic yard, cord.

Solids are of various shapes; those that the class will consider at present are of the shape of the chalk-box;

solids of that shape are called rectangular solids; when the length, width, and thickness are equal, the solid is called a *cube.*

Teach pupils the definition of cubic inch, cubic foot, cubic yard; show them solids of the volume of the first two.

RELATION BETWEEN CUBIC INCH AND CUBIC FOOT

1. Have pupils construct with small blocks a solid 1 ft. long, 1 ft. wide, and 1 in. high.

2. Have them state the dimensions.

3. Have them tell the volume in cubic inches.

4. Have them tell how many such solids, or layers, would be required to make a pile 1 foot high.

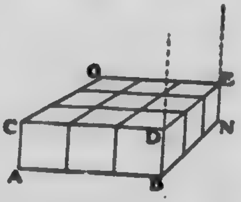
5. Have them tell the volume of this pile in cu. ft.

6. Lead them to calculate the volume in cubic inches—12 "layers", the volume of each of which is 144 cubic inches; and finally the relation between cubic feet and cubic yards.

RELATION BETWEEN CUBIC FOOT AND CUBIC YARD

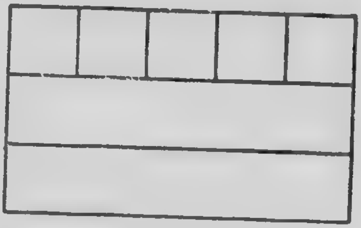
Unless the teacher has a sufficient number of suitable blocks for this purpose, he should draw a diagram on the black-board. ADNG represents a solid composed of a number of small cubes placed side by side; the dimensions of each of the cubes are 1 ft., by 1 ft., by 1 ft. Ask the pupils to tell the volume of each. Ask them to tell the dimensions of ADNG, and then its volume.

Lead them to ascertain the number of blocks like ADNG that would be required to make a pile 3 ft. high, 3 ft. long, and 3 ft. wide; have them compute its volume in cubic feet. Have them state its dimensions in yards, then its volume in yards, and finally the relation between cubic yard and cubic foot.



The greatest unit of volume, the cord, is equal to 128 cu. ft.

PROBLEMS IN MEASUREMENT



1. Find the area of a rectangle 5 in. long and 3 in. wide.

Let the oblong be divided into 3 strips by lines 1 inch apart as in the figure.

The area of one strip = 5 sq. in.

The area of the rectangle = 3 times 5 sq. in. = 15 sq. in.

In this figure there are two units of measurement. The smaller, a primary unit of area, is 1 sq. in. and is repeated five times to make the larger unit or the strip; the larger, or derived, unit of one strip is 5 sq. in. and is repeated three times to make the oblong which is now measured.

Make a rectangle 5 inches long and 3 inches wide. Divide it as in the figure, and make a mental picture of the resultant figure.

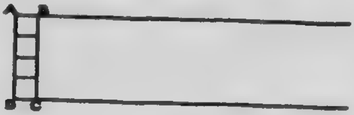
It will be observed that the *number* of units in the area is the product of the *numbers* that measure the length and width respectively in the corresponding linear units. Thus, the area of the rectangle just considered is 15 sq. in.; the number 15 was obtained by multiplying 5 by 3, the *numbers* that measure the dimensions of the rectangle.

Similarly, if the problem were to find the area of a rectangle 8 miles long by 4 miles wide, one would suppose the figure divided into 4 strips by lines drawn a mile apart, and one of the strips divided into 8 equal parts by drawing lines across it at a distance of 1 mile. Each of the small rectangles is 1 mile by 1 mile, and its area 1 sq. mi.; each strip contains 8 of these and, therefore, its area is 8 sq. mi.; the area of the whole rectangle is 4 times 8 sq. mi. or 32 sq. mi., the product of 8 and 4, the *numbers* that measure the dimensions in corresponding linear units.

The solution may be written thus:

Area of rectangle $=1$ sq. mi. $\times 8 \times 4 = 32$ sq. mi.

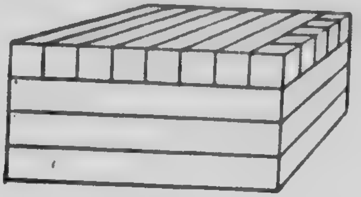
2. To find the length of a rectangle whose area is 36 sq. in. and whose width is 4 in.



Draw a figure like this one. The dimensions of the strip are 1 in. by 4 in., and its area 4 sq. in. Since 36 contains 4 nine times, the area of the given rectangle is 9 times as great as this one. Therefore, it would take 9 rectangles the size of ABCD placed side by side to make one equal in area to the given one; and 9 placed side by side would make a rectangle 9 in. long.

The problem might have been solved by remembering that since the area of a rectangle is obtained by finding the product of the numbers that measure the dimensions, the number that measures one dimension can be found by obtaining its co-factor with respect to the number that represents the area; for example, if the area of a rectangle be 36 sq. in. and the measure of one dimension be 4, that of the other dimension must be 9.

3. To find the volume of a rectangular solid 8 inches long, 5 inches wide, and 4 inches high.



Let the solid be divided into 4 slices by horizontal planes 1 inch apart. Let the right-hand row be divided into 5 cubic inches by vertical planes 1 inch apart.

The volume of 1 row=5 cubic inches.
The volume of 8 rows or 1 slice=5 cubic inches×8.
The volume of 4 slices=5 cubic inches×8×4,
=160 cubic inches.

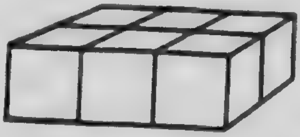
In this solid the three units of volume, in order of size, are the primary unit or 1 cu. in., the derived units or the row of 5 cu. in., and the slice of 40 cu. in.

The solid is made up of how many units of each kind?

Each unit is made up of how many of the next smaller?

It may be observed that the *number* that measures the volume of the solid is the product of the *numbers* that measure its dimensions. The *number* of cu. in. in the solid, 160, is the product of 4, 8, and 5, the measures of the dimensions.

4. Given the volume, and two of the dimensions of a rectangular solid, to find the third.



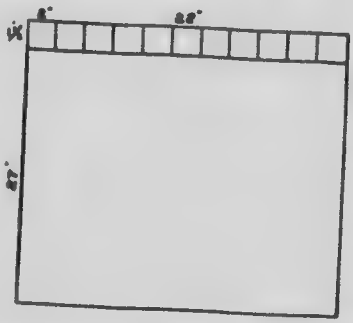
The volume of a rectangular solid is 24 cu. ft.; two of its dimensions are 2 ft. and 3 ft. respectively; find the third dimension.

The volume of a slice, 2 ft. by 3 ft. by 1 ft., is 6 cu. ft. The volume of the given solid is 24 cu. ft. or 4 times 6 cu. ft., that is, it is equal in volume to that of 4 such slices. A pile of four slices would make a solid 4 ft. high, therefore, 4 ft. is the third dimension.

The problem might also have been solved by remembering that the number that measures the volume is the product of the three numbers that measure the dimensions in corresponding linear units. The product of 2, 3, and the number that measures the third dimension is 24, and therefore the number that measures the third dimension is 24÷6, or 4.

OTHER PROBLEMS

1. How many cards 2 in. by 1½ in. can be cut from 50 sheets of bristol board, each sheet being 27 in. by 22 in.? What will the cards cost at 6c. a hundred?



The first step in the solution is to determine in what way the cards can be cut most economically from the sheet. Since 22 inches is an exact multiple of 2 inches, and 27 inches is not, it is plain that there will be no waste in that direction, if the length of the card be cut along the short side of the sheet.

Since 22 inches contains 2 in. 11 times, there will be 11 cards in a strip of bristol board $1\frac{1}{2}$ inches wide and 22 inches long; the number of such strips that can be cut from a sheet will be the number of times that $1\frac{1}{2}$ inches is contained in 27 inches, that is 18.

In 1 sheet there are 11 cards × 18.
In 50 sheets there are 11 cards × 18 × 50 or 9900 cards.

The teacher will note that this problem is not solved by dividing the area of the broad surface of the card into the area of the broad surface of a sheet of the bristol board.

2. Around a two-acre lot which is 20 rd. long, the owner builds a wire fence with posts 22 ft. apart. How many posts will be required?

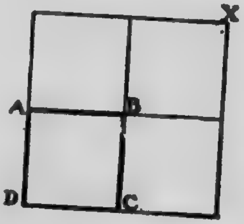
In this problem, its practical application must be considered. There must be a post at each corner of the field. The dimensions must be found as a first step in the solution; they are 16 rods and 20 rods. The pupil must first find the number of posts required for each side separately and add; but, in totalling the numbers, must remember that the post at each corner must be counted only once.

Since 16 rods and 20 rods are both multiples of 22 ft., this particular problem might have been solved by simply dividing the perimeter by 22 ft.

If a dimension of the rectangular field were not a multiple of the distance between the posts, what would be the number of posts required?

3. There are two square plots of ground, the length of one being twice that of the other. If it costs $20 to sod the smaller of the two, what will it cost at the same rate to sod the larger?

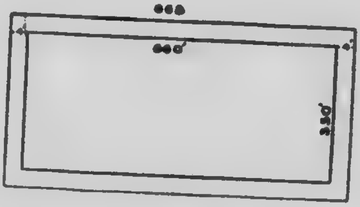
The problem here is to find the area of the larger plot in terms of the area of the smaller, or to find how many times the area of the small plot is contained in the area of the large plot. This is easily done by drawing a diagram.



Let A B C D represent the smaller plot; produce each of its sides D A and D C, the length of itself and complete the square D X; then produce the sides A B and C B until they meet the sides of the large square. The larger square, D X, represents the large plot, since each of its sides is double the length of the sides of A C. It is seen from a glance at the diagram, that the large square is four times as great as the small one and that, therefore, the cost of sodding it will be four times $20.

The pupils should be led to draw the diagram and make the observation.

4. Around the outside of a block of land 40 rd. by 20 rd. there is a cement walk 4 ft. wide. What did the walk cost at 15c. a sq. ft.?



Draw a diagram to scale and show the walk consists of two rectangles 668 ft. by 4 ft. and two rectangles 330 ft. by 4 ft. Have the pupils draw all diagrams if possible.

The principle taught in this problem is useful in solving a number of other problems; for example, finding the volume of the walls of a rectangular house; finding the dimensions of the panes of glass in a window-sash when the outside dimensions of the sash and the thickness of the wood are given.

In solving problems in measurement it will be found helpful to draw diagrams to a scale.

SHINGLING

Shingles are sold by the thousand. They are put up in bunches usually of 250 shingles each. There are, therefore, 4 bunches in a thousand. The bunch is about 22 inches wide and usually contains 25 layers at each end, or 50 layers in all. Any number of bunches may be bought, but not less than 1 bunch. A standard shingle is 4 inches wide, though the shingles in a bunch are often more or less than 4 inches, and are therefore more or less than 1 standard shingle. Shingles are laid in rows from end to end on a roof. These rows vary in width according to the number of inches of the row "to the weather"—which may be 4 inches, 4½ inches, or 5 inches. The first row at the eaves must be double.

In order to make allowance for defective shingles and for the fact that shingles are placed more closely together on a roof than in the bunch, builders usually estimate that a bunch laid 4 inches to the weather will cover 25 square feet of roof or a thousand shingles will cover 100 square feet of roof.

If shingles are placed 5 inches to the weather, an inch more of the length of each shingle will be exposed than if it were laid 4 inches to the weather, therefore it will cover one fourth more surface.

Illustration

Take this problem:

A double roof is 50 ft. long with rafters 20 ft. What will it cost to shingle this roof with standard shingles, if the shingles are laid 5 in. to the weather and cost $3.00 per thousand?

Length of rafter $=20$ ft. $=240$ in.
Width of row $=5$ in.
Number of rows on each side $=(240\div 5)+1=49$,
∴ Number of rows on both sides $=49$ rows $\times 2=98$ rows.
Length of rows $=50$ ft.
There are 3 shingles in 1 ft. wide.
Number of shingles in row $=3$ shingles $\times 50=150$ shingles.
Number of shingles on roof $=150$ shingles $\times 98=14700$ shingles.
∴ We must pay for 14 thousand, and 3 bunches.
Cost $=\$3.00\times 14\frac{3}{4}=\$44.25$.

In problems like the following the shingles referred to are of particular sizes, such as one may sometimes see on railway stations or on fancy roofs.

1. If a shingle covers 4 in. by 3½ in. of a roof, how many shingles will be required for the two sides of a roof of a barn, each side of the roof being 35 ft. by 20 ft.?

2. If shingles are 4 in. wide and are laid 5 in. to the weather, how many hundreds will be required for a roof 40 ft. long and 16 ft. 8 in. from eaves to ridge?

These questions simply ask us to find how many times a large rectangular area (a side of the roof) contains a smaller rectangular area (the flat side of a shingle).

BOARD MEASURE

1. How much lumber $1\frac{1}{2}$ in. in thickness will it require to build a board sidewalk 4 ft. wide and $\frac{1}{8}$ of a mile long, the boards being nailed to 3 scantlings 4 in.×4 in.?

The area of sidewalk is 1 sq. ft.$\times 4\times\frac{1}{8}\times 5280$.

The number of board feet is $1\frac{1}{2}$ times the number expressing the area in sq. ft. (See *Ontario Public School Arithmetic,* page 121.)

$$\begin{aligned}\text{Number of board ft.} &= 1\tfrac{1}{2}\times(4\times\tfrac{1}{8}\times 5280),\\ &= \tfrac{3}{2}\times 4\times\tfrac{1}{8}\times 5280,\\ &= 3960.\end{aligned}$$

The area of one side of a scantling the length of the sidewalk is $(\frac{4}{12}\times\frac{1}{8}\times 5280)$ sq. ft. The number of board ft. is 4 times the number of sq. ft. in the area of the side of the scantling.

$$\begin{aligned}\text{Total number of board ft. in scantlings}\\ &= 3\times 4\times\tfrac{4}{12}\times\tfrac{1}{8}\times 5280,\\ &= 2640.\end{aligned}$$

Amount of lumber in all$=(2640+3960)$ ft.$=6600$ ft.

2. What will be the cost of inch maple for a hardwood floor for a room 25 ft. by 20 ft. at $65 per M?

The number of board feet of lumber required is the same as the number of square feet in the area.

Hardwood flooring is usually either $\frac{3}{8}$ inches or $\frac{7}{8}$ inches in thickness, is always tongued and grooved, and cut into strips or boards of uniform width. The lumber is cut in the saw-mill into boards $2\frac{1}{2}$ inches or 3 inches wide, and then taken to the planing-mill and dressed—that is, planed, tongued, and grooved; about $\frac{1}{8}$ inch is taken off the thickness in planing and ½ inch off the width by the tongue, so that a board that was 3 inches wide originally will cover a strip of floor 2½ inches wide.

3. What will be the cost of hardwood flooring ⅞ in. thick, cut in boards 3 in. wide and tongued and grooved, for a room 25 ft. by 20 ft. at $65 per **M**?

A board 3 inches wide will cover a strip of floor 2½ inches wide; in other words, 2½ inches of the board's width is used as a walking surface and ½ inch runs into the groove in the next board. When the floor is laid, ⅕ of the width of "walkable" surface of a board is hidden from view. The buyer pays for the whole board.

The number of board ft. required is ⅕ more than the number of sq. ft. in the surface of the floor.

The solution may be written this way:

$$\frac{\$65\times25\times20\times6}{1000\times5}=\text{cost of lumber.}$$

CHAPTER XV

FRACTIONS—PART I

SIMPLE FRACTIONAL UNITS

OBJECTIVE TREATMENT

MOST children have a usable knowledge of the meaning of one half and one quarter when they enter school. With the aid of objective material the teacher can make this knowledge more accurate and make clear the meanings of such other fractional parts as one eighth, one third, one sixth, one ninth, one fifth, one tenth, and one seventh.

ONE HALF

Divide into any two parts some object, such as an apple, a crayon, a sheet of cardboard or drawing paper, a bundle of kindergarten sticks, etc. Then ask pupils which part they would like to have and why. Next let them suggest how the object might have been divided into two parts so that there would have been no choice as to the part they would like to have. Then tell them that when anything is divided into *two equal* parts it is said to be divided into *halves* and each of the parts is called *one half* of the thing divided. Now give practice in dividing into halves such objects as strips of paper, strings, sheets of folding paper of various shapes and sizes, etc. Let the pupils suggest different ways in which the division can be made. Let them compare the size of the parts made by any one division with those obtained from a different

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division. They can then be given a cord and asked to find the half of the length of their desks, of the teacher's desk, and half the width of the room.

Each pupil may take his ruler and point out one half of it—that is, one half a foot—and tell how many inches there are in one half of one foot. Pupils may also point out one half of an inch and tell how many half-inches there are in one inch, two inches, two and one-half inches, etc.

They may take objects and find out one half of two, four, six, etc., then one half of three, five, seven, etc. They may next find one half of a yard, one half of a quart, or of such other measures as have already been taught. They may determine by addition the whole of any quantity of which the half is known. One half of a number is 2, what is the whole number? How many halves are there in one whole? If from one whole one half is taken away what is left?

ONE FOURTH

Give each pupil a sheet of folding paper or a strip of cover paper and ask him to fold it into halves. Now fold each of the halves into halves. Each of these is what part of one half? Into how many parts has the whole sheet of paper been folded? Which of these parts is the largest? How do you know? Into how many equal parts has the paper been folded?

When anything is divided into four equal parts it is said to be divided into *fourths,* and each of the equal parts is called *one fourth* of the whole thing divided. Point out one fourth of the paper, two fourths, three fourths. How many fourths are there in one whole? Which is the larger, one half or one fourth? Find one half of your paper. How many fourths are there in this? One half is then the same as how many fourths? How much of the paper is there in one half and one fourth of it? Find different ways in which the paper can be folded into fourths. Try to find if any of the fourths are larger than any of the others. Now fold pieces of paper, cards, and other objects into fourths.

Take your ruler and find one fourth of an inch. It is what part of one half of an inch? How many fourths are there in two inches? In two and one-fourth inches? In three and one-half inches? How many inches are there in one fourth of a foot? Take up eight blocks and show how to find one fourth of them. With blocks, make up the number in the one fourth of which there are three blocks.

ONE EIGHTH

One eighth can be found by taking one half of one fourth or one fourth of one half, and on it the pupils may be given exercises similar to those above. They may also be asked to give different names for one half and one fourth and to find by using folding paper:

1. The sum of one half and one eighth; three fourths and one eighth; one half, one fourth, and one eighth;
2. The difference between one half and one eighth; one fourth and one eighth, etc.

ONE THIRD

Knowing that quantities may be divided into two, four, and eight equal parts, the pupils might suggest that it is also possible to divide them into three equal parts. Let them endeavour to do this with folding paper, with blocks, with their rulers, etc. They may then be told that when anything is divided into three equal parts it is said to be divided into *thirds,* and each of the parts is said to be *one third* of the thing divided.

Exercises, similar to those given in other fractions, will follow. Give also such exercises as: which is the greater, one half or one third? one third or one fourth? With blocks build up the number which contains three in its one third.

ONE SIXTH AND ONE NINTH

These fractions can be obtained by taking one half of one third and one third of one third. Exercises should be given to find the *sixth* and *ninth* of quantities, to build up a number whose sixth and ninth is given, to make comparisons with other fractions taught, to add or subtract such fractions as one half and one third, and to give different names for fractions such as one half, two thirds, etc.

ONE FIFTH, ONE TENTH, ONE SEVENTH

These fractions should be dealt with in a manner similar to that used for one third. One *tenth* will be obtained by finding one half of one *fifth* or one fifth of one half.

The pupils may now be shown how to write the fractions. Thus: ½, ¼, ¾, ⅙, ⅚, etc.

FRACTIONS—PART II

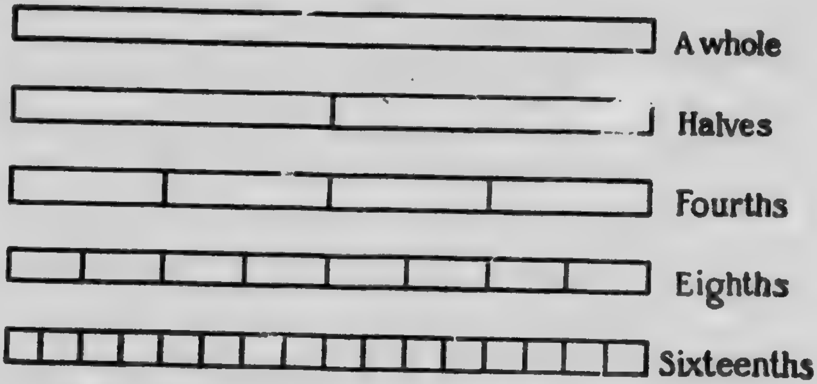
DEFINITIONS, TERMS, OPERATIONS

SYSTEMATIC TREATMENT

Give to each member of the class 5 strips of thin cardboard or cover paper one inch wide and 8 inches long. The teacher may use 5 strips 16 inches long. Let each pupil note that the strips he has are all the same length.

Let each pupil put one strip on the table. Take the second strip and, after folding into two equal parts, cut the strip at the crease and set down parallel to the first strip. Take the third strip, fold twice, and cut into 4 equal parts, and set down parallel to the second. Similarly, by folding, divide the fourth strip into 8 equal parts, and the fifth into 16 equal parts, and set down as before.

The following diagram will represent the division and disposition of the strips:



Important.

In the first strip we have a whole, or a prime, unit.

In the second strip how many equal parts have we? Two. What shall we call these parts? Halves.

In the third strip how many equal parts have we? Four. What shall we call these parts? Fourths.

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In the fourth and fifth strips respectively how many equal parts have we? Eight and sixteen. What shall we call these parts respectively? Eighths and sixteenths.

Now any one or more of the equal parts into which the whole, or prime, unit, is divided is a *fraction.*

Thus, in the fourth strip, one part, or one eighth, is a fraction; two parts—that is, two eighths, is a fraction; three parts—that is, three eighths, is a fraction; etc., etc.

In the fourth strip what is one of the equal parts called? One eighth. What are seven of these equal parts called? Seven eighths or 7 times one eighth.

The fraction "seven eighths" may be regarded as a quantity got by repeating a unit of measurement, in this case a fractional unit, "one eighth", 7 times, just as 7 feet is a quantity got by repeating the unit of measurement, "1 foot", 7 times.

NOTATION

Take 7 of the equal parts of the fourth strip and we have the fraction "seven eighths", which might be written 7 eighths or 7 times one eighth, but is generally written $\frac{7}{8}$.

In the fifth strip what shall we call nine of the equal parts? Nine sixteenths. How shall we write this? 9 times one sixteenth, or $\frac{9}{16}$.

What is it that determines the fractional unit, or the unit of measurement—that is, the name of the parts, in the case of each of the strips which are divided? Where, for example, in $\frac{9}{16}$, does this number appear? Under the line. Because this number determines the name, or denomination, of the fraction it is called the *denominator,* or "name teller", of the fraction.

And because the number above the line tells the number of the equal parts taken, or the number of times the unit of measurement (in this case, one sixteenth) is repeated, it is called the *numerator,* or "number teller", of the fraction.

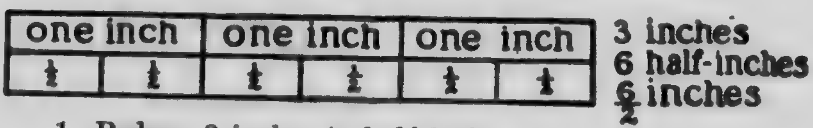
The denominator and numerator are the *terms* of a fraction. (See *Ontario Public School Arithmetic,* pages 92-4.)

NUMERATION

Take strips of paper, say 6 inches long, and show the meaning of $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{6}$, $\frac{3}{12}$, $\frac{1}{4}$, $\frac{3}{4}$. (See *Ontario Public School Arithmetic,* page 87. Examples 6, 7, 8.)

REDUCTION

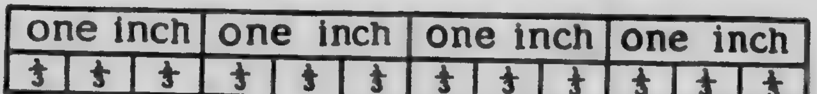
To change a whole number to a fraction



1. Reduce 3 inches to half inches.

Analysis: 1 in. = 2 half-inches,
3 in. = 3×2 half-inches,
= 6 half-inches,
= $\frac{6}{2}$ inch.

2. Reduce 4 inches to thirds of an inch.



4 in. = 12 one-third in.

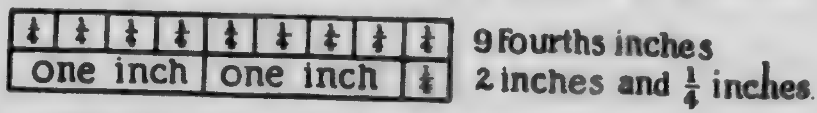
Analysis: 1 in. = 3 thirds of an inch,
4 in. = 4×3 thirds of an inch,
= 12 thirds of an inch,
= $\frac{12}{3}$ inches.

3. Reduce 7 inches to sixths of an inch.

Brief analysis:

1 in.$=$6 sixths of an inch$=\frac{6}{6}$ in.,
7 in.$=7\times6$ sixths of an inch$=42$ sixth-in.$=\frac{42}{6}$ in.

To change a whole number and a fraction to a fraction



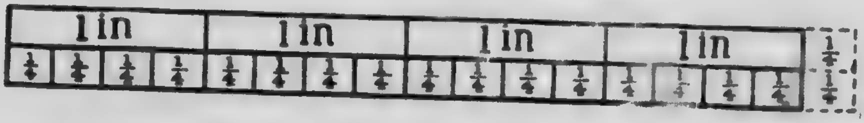
Example: Reduce $2\frac{1}{4}$ inches to fourth-inches.

Analysis:

1 in.$=4$ fourth-in.,
2 in.$=2\times4$ fourth-in.$=\frac{8}{4}$ in.,
2 in.$+1$ fourth-in.$=8$ fourth-in.$+1$ fourth-in.$=\frac{9}{4}$ in.

or briefly, thus: $2\frac{1}{4}$ in.$=\frac{2\times4+1}{4}$ in.$=\frac{9}{4}$ in.

To change a fraction to a whole number



Example: Change $\frac{17}{4}$ in. to in.

4 in. and $\frac{1}{4}$ in. or $4\frac{1}{4}$ in.$=\frac{17}{4}$ in.

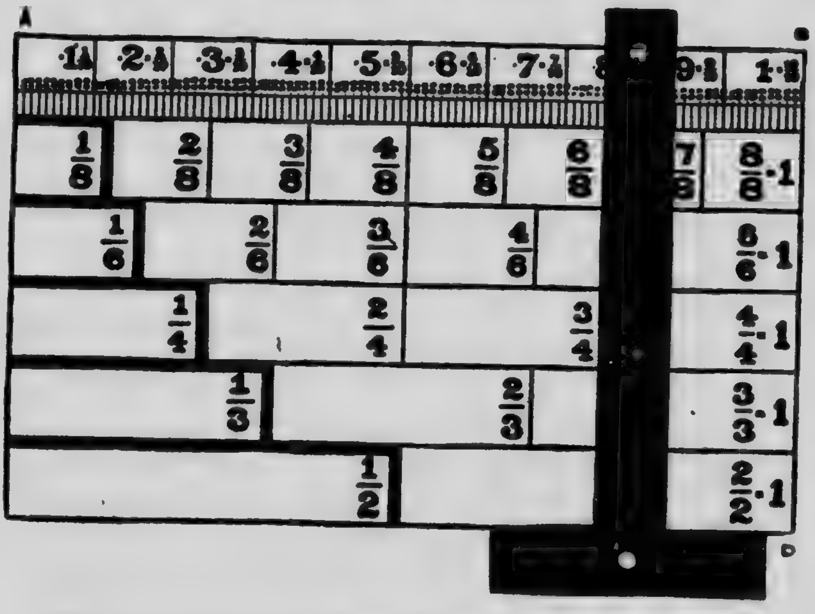
Analysis:

4 fourth-inches$=1$ inch,
16 fourth-inches$=4$ inches,
16 fourth-inches$+1$ fourth-inch$=4$ inches$+1$ fourth-inch,
$\frac{17}{4}$ inches$=4\frac{1}{4}$ inches.

or briefly: $\frac{17}{4}$ inches$=(17\div4)$ in.$=4\frac{1}{4}$ in.

Reduction of fractions to equivalent fractions

Outline on the black-board a rectangle 48 in. long and 10 in. wide, and divide as in the following diagram:



Write down therefrom equivalents for:

$\frac{1}{2} = \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$.
$\frac{1}{3} = \frac{2}{6}, \frac{3}{9}$.
$\frac{1}{4} = \frac{2}{8}, \frac{3}{12}$.

$\frac{3}{4} = \frac{6}{8}, \frac{9}{12}$.
$\frac{2}{3} = \frac{4}{6}, \frac{6}{9}$.

By comparing any pair of equivalent fractions, lead the class to observe that if both numerator and denominator of a fraction be multiplied or divided by the same number the fraction is not altered in value.

By comparing the eighths and the fourths in the diagram above we see:

1. That in the whole line there are $\frac{8}{8}$, or 8 times the fractional unit, $\frac{1}{8}$.

2. That in the whole line there are $\frac{4}{4}$, or 4 times the fractional unit, $\frac{1}{4}$.

3. That the fractional unit, $\frac{1}{4}$ = twice the fractional unit, $\frac{1}{8}$.

4. That 2 times $\frac{1}{4}$ = 4 times $\frac{1}{8}$, that is, that when the fractional unit of measurement is made one half as great, it must be taken twice as many times. (See *Ontario Public School Arithmetic,* pages 92, 93, 94.)

ADDITION AND SUBTRACTION OF FRACTIONS

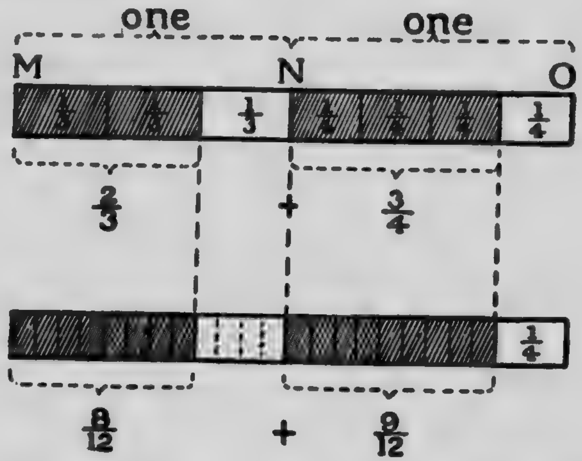
$$\text{Add (1)}\quad \begin{array}{r}6\text{ ft.}\\ \underline{8\text{ in.}}\end{array}\qquad\qquad (2)\quad \begin{array}{r}2\text{ bu.}\\ \underline{3\text{ gal.}}\end{array}$$

$$\text{Here we change (1) from}\quad \begin{array}{r}6\text{ ft.}\\ \underline{8\text{ in.}}\end{array}\quad\text{to}\quad \begin{array}{r}72\text{ in.}\\ \underline{8\text{ in.}}\end{array}\quad\text{or}\quad \begin{array}{rl}72&(1\text{ in.})\\ \underline{8}&(1\text{ in.})\\ 80&(1\text{ in.})\end{array}$$

$$\text{and (2) from}\quad \begin{array}{r}2\text{ bu.}\\ \underline{3\text{ gal.}}\end{array}\quad\text{to}\quad \begin{array}{r}16\text{ gal.}\\ \underline{3\text{ gal.}}\end{array}\quad\text{or}\quad \begin{array}{rl}16&(1\text{ gal.})\\ \underline{3}&(1\text{ gal.})\\ 19&(1\text{ gal.})\end{array}$$

These examples illustrate the fact that before adding whole numbers we must reduce the addends to the same denomination, or express each as a certain number of times the same unit of measurement. So, also, fractions can be added only when they are of the same denomination, or are expressed as so many times the same fractional unit.

Add $\frac{2}{3}$ and $\frac{3}{4}$.



In the diagram MO, MN, and NO, each equals one whole.

Let the pupils note:

1. That $\frac{2}{3} = \frac{8}{12} = 8$ twelfths,

2. That $\frac{3}{4} = \frac{9}{12} = 9$ twelfths,
and that $\therefore \frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$;

3. That 12 is the smallest number that will contain 3 and 4 exactly, and therefore the smallest denominator that may be used for equivalent fractions, and that therefore $\frac{1}{12}$ is the greatest unit of measurement that can be used.

This may be shown thus:

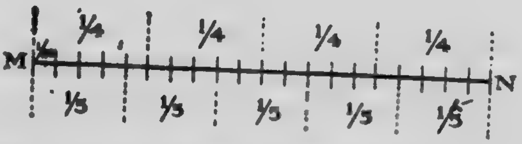
$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{14}{21} = \frac{16}{24} = \text{etc.}$$
$$\frac{3}{4} = \frac{6}{8} \qquad = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} \qquad = \frac{18}{24} = \text{etc.}$$

Hence $\frac{2}{3}$ and $\frac{3}{4}$ may be expressed as $\frac{8}{12}$ and $\frac{9}{12}$, or as $\frac{16}{24}$ and $\frac{18}{24}$; that is, as $8(\frac{1}{12})$ and $9(\frac{1}{12})$, or as $16(\frac{1}{24})$ and $18(\frac{1}{24})$.

The greatest unit of measurement ( ) is the one generally used. As will be shown later, the denominator 12 is, of course, the L.C.M. of 3 and 4, that is, it is the least number that will contain both 3 and 4.

ORAL EXERCISES

A



1. How many are there of the smallest parts of MN? Each of them is what part of the line?

2. $\frac{1}{5}$ = how many twentieths? $\frac{1}{4}$ = how many twentieths?

To what may we change fourths and fifths, or, in other words, what unit of measurement may we use in order to find their sum?

3. 4 × 5 = ? 20 = 5 × ? 20 ÷ 4 = ?
What is 20 of 4 and 5?

$\frac{2}{5} = \frac{?}{20}$; $\frac{3}{5} = \frac{?}{20}$; $\frac{4}{5} = \frac{?}{20}$; $\frac{2}{4} = \frac{?}{20}$; $\frac{3}{4} = \frac{?}{20}$.

4. $\frac{1}{4} + \frac{1}{20} = ?$

5. $\frac{1}{5} + \frac{1}{20} = ?$

6. $\frac{3}{4} + \frac{1}{20} = ?$

7. $\frac{4}{5} + \frac{1}{20} = ?$

8. $\frac{1}{4} + \frac{1}{5} = ?$

9. $\frac{2}{5} + \frac{1}{4} = ?$

10. $\frac{1}{4} + \frac{3}{5} = ?$

11. $\frac{4}{5} + \frac{1}{4} = ?$

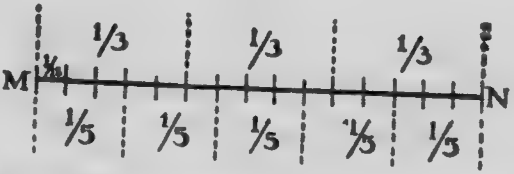
12. $\frac{3}{4} + \frac{1}{5} = ?$

13. $\frac{2}{5} + \frac{3}{4} = ?$

14. $\frac{3}{5} + \frac{2}{4} = ?$

15. $\frac{3}{4} + \frac{4}{5} = ?$

B



1. The line MN is divided into how many equal parts? What is each of these parts?

2. $\frac{1}{5}$ = how many fifteenths? $\frac{1}{3}$ = how many fifteenths?

3. $\frac{2}{5}=\frac{?}{15}$; $\frac{3}{5}=\frac{?}{15}$; $\frac{4}{5}=\frac{?}{15}$; $\frac{2}{3}=\frac{?}{15}$.

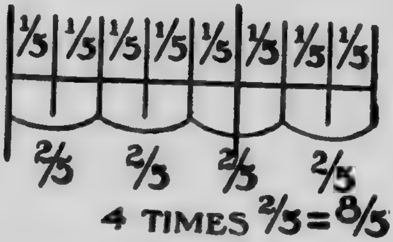
4. $3 \times 5 = ?$ $15 \div 3 = ?$ $15 \div 5 = ?$ 15 is what of 3 and 5?

5. To what shall we change thirds and fifths so that we may add them or subtract them?

6. $\frac{1}{3}+\frac{1}{15}=?$
7. $\frac{1}{5}+\frac{1}{15}=?$
8. $\frac{1}{3}+\frac{1}{5}=?$
9. $\frac{1}{3}+\frac{2}{5}=?$
10. $\frac{1}{5}+\frac{2}{3}=?$
11. $\frac{2}{5}+\frac{1}{3}=?$
12. $\frac{1}{3}+\frac{4}{5}=?$
13. $\frac{2}{3}+\frac{2}{5}=?$
14. $\frac{2}{3}-\frac{2}{5}=?$
15. $\frac{1}{5}-\frac{1}{15}=?$
16. $\frac{1}{3}-\frac{2}{15}=?$
17. $\frac{1}{3}-\frac{1}{5}=?$

MULTIPLICATION AND DIVISION OF FRACTIONS

To multiply a fraction by a whole number



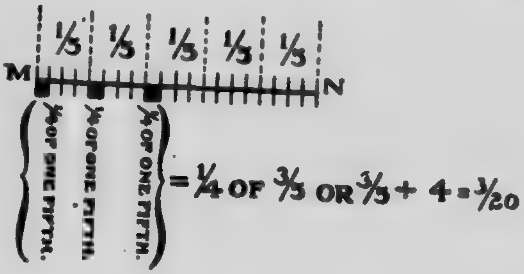
Example: $\frac{2}{5} \times 4 = \frac{8}{5}$.
This is easily seen from the diagram.
It may be represented thus:
2 fifths × 4 = 8 fifths,
$\frac{2}{5} \times 4 = \frac{2 \times 4}{5} = \frac{8}{5}$.

From this and other examples derive the rule: *To multiply a fraction by a whole number multiply the numerator of the fraction by that number and retain the denominator.* (See *Ontario Public School Arithmetic*, pages 99-100.)

To divide a fraction by a whole number

Example: $\frac{3}{5} \div 4$.

This means to divide $\frac{3}{5}$ into 4 equal parts and find the value of each of these parts. In the diagram the line MN is first divided into fifths; then each of these fifths is divided into four equal parts; next, one of these equal parts of each of the three fifths is taken. It is evident that $\frac{1}{4}$ of $\frac{3}{5}$, that is, $\frac{3}{5} \div 4$, is equal to $\frac{3}{20}$, since the small parts are twentieths.



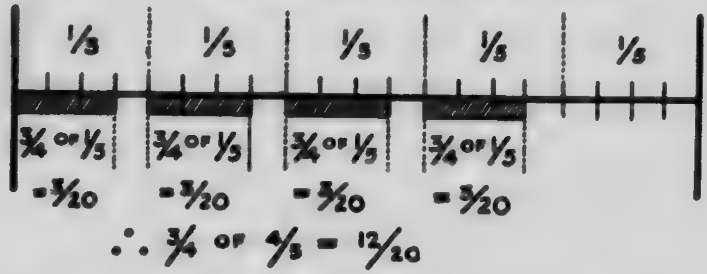
Again: $\frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20}$, as shown in a foregoing exercise.

Hence, to divide $\frac{3}{5}$ by 4 is to divide $\frac{12}{20}$ by 4, and 12 twentieths ÷ 4 is 3 twentieths or $\frac{3}{20}$.

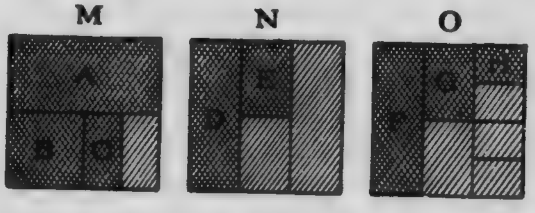
From this and similar examples we infer the rule that *a fraction is divided by a whole number when its denominator is multiplied by that number.*

To find the value of a fraction of a fraction, that is, of a compound fraction

1. See *Ontario Public School Arithmetic,* page 101.
2. See following diagram to show $\frac{3}{4}$ of $\frac{4}{5}=\frac{12}{20}$.



3. In figure M, what part of M is A? $\frac{1}{2}$ of M. What part of A is B? $\frac{1}{2}$ of B. Then what part of M is B? $\frac{1}{2}$ of



$\frac{1}{2}$ of M. Looking only at B and M, what part of M is B? $\frac{1}{4}$. Then $\frac{1}{2}$ of $\frac{1}{2}=\frac{1}{4}$.

Again, C is what part of B? $\frac{1}{2}$. And B is what part of M? $\frac{1}{4}$. Then C is what part of M? $\frac{1}{2}$ of $\frac{1}{4}$. Looking only at C and M, what part of M is C? $\frac{1}{8}$. Then $\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$.

In figure N say in two ways what part E is of N.

(*a*) $\frac{1}{2}$ of $\frac{1}{3}$, or (*b*) $\frac{1}{6}$,
$\therefore$ $\frac{1}{2}$ of $\frac{1}{3}$ $=\frac{1}{6}$.

Show by figure O that $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{3}=\frac{1}{12}$.

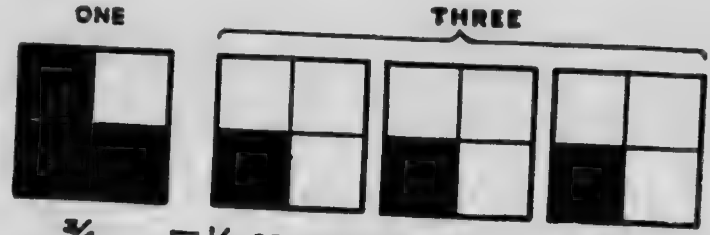
See now diagram, *Ontario Public School Arithmetic,* page 101.

To Multiply a Fraction by a Fraction

Example: $\frac{2}{5} \times \frac{3}{4}$.

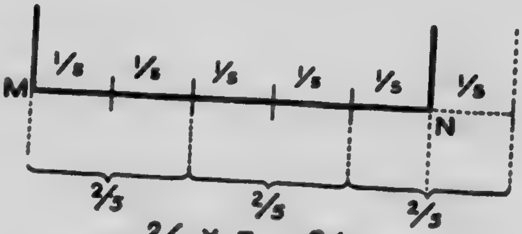
In this operation three steps are necessary:

1. To show that the multiplier $\frac{3}{4} = \frac{1}{4}$ of 3



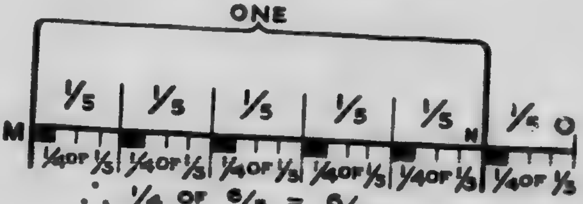
$\frac{3}{4}$ = 1/4 OF ONE + 1/4 OF ONE + 1/4 OF ONE
= 1/4 OF THREE
= 1/4 OF 3

2. To multiply $\frac{2}{5}$ by 3



$\frac{2}{5} \times 3 = \frac{6}{5}$

3. Here we have multiplied by 3 instead of $\frac{1}{4}$ of 3, and we must, therefore, take $\frac{1}{4}$ of the result $\frac{6}{5}$.



∴ $\frac{1}{4}$ OF $\frac{6}{5} = \frac{6}{20}$.

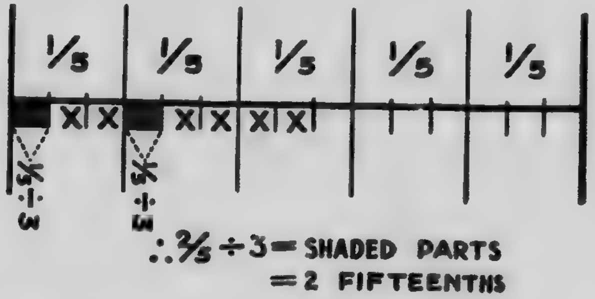
MN is 1 and MO $= \frac{6}{5}$ of 1,
MN is divided into 20 equal parts,
$\therefore$ each part is $\frac{1}{20}$.
$\frac{1}{4}$ of each of the 6 fifths $= \frac{1}{20}$,
$\therefore \frac{1}{4}$ of $\frac{6}{5} = \frac{6}{20}$,
$\therefore \frac{6}{5} \times \frac{1}{4} = \frac{6}{20}$.

From this and similar examples we infer the rule for the multiplication of two fractions: *The product of two fractions is equal to the fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the given fractions.*

To Divide a Fraction by Another Fraction

Example: Divide $\frac{2}{5}$ by $\frac{3}{4}$.

Since $\frac{3}{4} = \frac{1}{4}$ of 3, if we divide $\frac{2}{5}$ by 3 instead of by $\frac{1}{4}$ of 3, we must multiply our result by 4.



In the diagram we have shown $\frac{2}{5} \div 3$ as giving two of the small parts called fifteenths, and these two parts are taken 4 times, making 8 fifteenths.

Therefore $\frac{2}{5} \div \frac{3}{4} = \frac{8}{15} = \frac{2 \times 4}{5 \times 3} = \frac{2}{5} \times$ divisor inverted, Or otherwise, thus:

$\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \div$ ($\frac{1}{4}$ of 3),

$= \frac{2}{5} \div$ 3 and the result multiplied by 4,

$= (\frac{2}{5 \times 3}$ or $\frac{2}{15})$ to be multiplied by 4,

$= \frac{2 \times 4}{15} = \frac{8}{15} = \frac{2 \times 4}{5 \times 3}$,

$= \frac{2}{5} \times \frac{4}{3}$. Now $\frac{4}{3}$ is called the reciprocal of $\frac{3}{4}$.

From this and similar examples we infer the rule: *To divide one fraction by another we make the reciprocal of the divisor and proceed as in the multiplication of fractions.*

Otherwise: $\frac{2}{5} \div \frac{3}{4} = \frac{8}{20} \div \frac{15}{20}$,

= 8 twentieths ÷ 15 twentieths = 8 ÷ 15,

$= \frac{8}{15} = \frac{2 \times 4}{5 \times 3}$. Hence the rule.

CHAPTER XVI

MEASURES AND MULTIPLES

DIVISORS, OR UNITS OF MEASUREMENT

UNIT OF MEASUREMENT

Let the class find all the lengths which may be used to measure exactly 12 in., 15 in., 18 in., 20 in.

Thus for 18 in. we have 1 in., 2 in., 3 in., 6 in., 9 in., 18 in.
" 12 " " 1 in., 2 in., 3 in., 6 in., 12 in.
" 15 " " 1 in., 3 in., 5 in., 15 in.
" 20 " " 1 in., 2 in., 4 in., 5 in., 10 in., 20 in.;
and 1 in., 2 in., 4 in., 5 in., 10 in., 20 in., are each Units of Measurement by which 20 in. may be exactly measured.

A Unit of Measurement, then, is a quantity by which another like quantity may be exactly measured.

COMMON UNIT OF MEASUREMENT

In the above, the units of measurement of 18 in., and 12 in., are given thus: for 18 in. we have 1 in., 2 in., 3 in., 6 in., 9 in., 18 in.; and for 12 in. we have 1 in., 2 in., 3 in., 6 in., 12 in. Of these 1 in., 2 in., 3 in., 6 in., are common, therefore each is a common unit of measurement of 12 in. and 18 in.

A Common Unit of Measurement of two quantities is such a quantity as may be used to measure exactly each of the two given quantities.

THE GREATEST COMMON UNIT OF MEASUREMENT

The common units of measurement of 12 in. and 18 in. are 1 in., 2 in., 3 in., and 6 in.

Of these 6 in. is the greatest, and it is therefore the Greatest Common Unit of Measurement of 12 in., and 18 in.

The Greatest Common Unit of Measurement of two or more quantities is the greatest quantity of the same kind by which each of these quantities may be exactly measured.

The Greatest Common Unit of Measurement may be called the *Greatest Common Measure.* If the quantities are abstract, or numbers, the Greatest Common Measure may be called the *Highest Common Factor.*

TO FIND THE GREATEST COMMON MEASURE

1. By trial

This method is illustrated in the foregoing sections.

2. By using prime factors

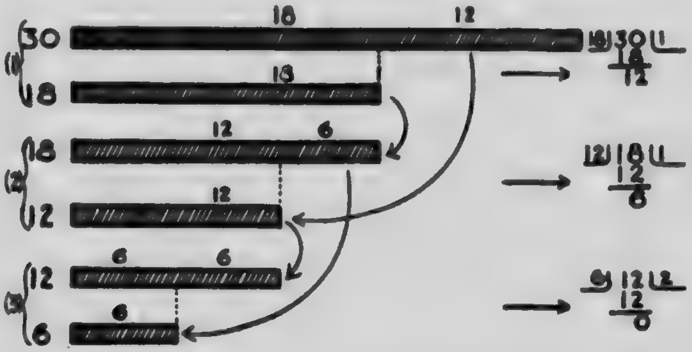
Find the Greatest Common Measure of 24 and 90. Factor each into its prime factors and set down thus:

$$24 = 2 \times 2 \times 2 \times 3$$
$$90 = 2 \times 3 \times 3 \times 5$$

Now, by inspection, it is seen that 2 and 3 are common divisors and, therefore 2×3, or 6 is the Greatest Common Measure.

3. By division

Find the Greatest Common Unit of Measurement of 18 in., and 30 in.



Greatest Common Unit of Measurement of 18 in. and 30 in. is 6 in.

1. Since the unit of measurement must measure 18 in. exactly, that is, turn over on 18 in. an exact number of times, and also measure 30 in. exactly, that is, turn over on 30 in. an exact number of times, (otherwise it would not be a unit of measurement) therefore it must turn over on the difference between 30 in. and 18 in., that is, on 12 in. an exact number of times, or measure it exactly.

2. Again, since it measures 18 in. exactly and 12 in. exactly it must measure their difference, namely 6 in. exactly.

3. And since it measures 12 in. exactly and 6 in. exactly it must measure their difference, namely 6 in., and the Greatest Common Unit of Measurement of 6 in. and 6 in. is clearly 6 in.

Therefore the Greatest Common Unit of Measurement for 30 in. and 18 in. is 6 in.

The principle illustrated above may be stated as follows: *The Greatest Common Unit of Measurement of any two numbers is a unit of measurement of their difference, or of the difference between any multiples of these numbers.*

Illustration

Find the G. C. U. of M. of 1643 and 6107.

1.	1643)6107(3 4929 1178	The Greatest Common Unit of Measurement of 6107 and 1643 is a unit of measurement of 6107 — (3×1643), or 1178. The Greatest Common Unit of Measurement is, therefore, to be found in 1643 and 1178.
2.	1178)1643(1 1178 465	If found in 1643 and 1178, it is found in their difference, which is 465, and, therefore, to be found in 1178 and 465.
3.	465)1178(2 930 248	If found in 1178 and 465, it is found in 1178—(2×465), that is, 248, and therefore, to be found in 465 and 248.
4.	248)465(1 248 217	If found in 465 and 248, it is found in their difference, which is 217, and, therefore, to be found in 248 and 217.
5.	217)248(1 217 31	Similarly, it is found in 248 — 217, that is, 31.
6.	31)217(7 217	It is found in 31 and 217 or 7×31, therefore the Greatest Common Unit of Measurement is 31.

The work may be formally set down as follows:

1643	3	6107
		4929
1178	1	1178
465	2	930
248	1	248
217	1	217
217	7	31

Therefore, the Greatest Common Unit of Measurement of 1643 and 6107 is 31.

If there be more than two quantities whose Greatest Common Unit of Measurement is required, find the Greatest Common Unit of Measurement of two, then of the result and the third, and so on.

MULTIPLES

MULTIPLE

Since 4 ft. ×1=4 ft.; 4 ft. ×2=8 ft.; 4 ft. ×3=12 ft.; 4 ft. ×4=16 ft.

Therefore 4 ft., 8 ft., 12 ft., 16 ft. contain 4 ft. an exact number of times, and are therefore said to be multiples of 4 ft.

Similarly 21, 28, 35, etc., are multiples of 7. Thus, *a Multiple of a given number, or quantity, is such a number, or quantity, as contains the given number, or quantity, a whole number of times.*

COMMON MULTIPLE

3, 6, 9, 12, 15, 18, 21, 24, are multiples of 3; 4, 8, 12, 16, 20, 24, are multiples of 4.

Examining the multiples we find that 12 and 24 are multiples of both 3 and 4 and are, therefore, Common Multiples of 3 and 4.

Also 2 ft., 4 ft., 6 ft., 10 ft., 12 ft., are multiples of 2 ft.; and 3 ft., 6 ft., 9 ft., 12 ft., are multiples of 3 ft.

Therefore, either 6 ft. or 12 ft. is seen to be a Common Multiple of 2 ft. and 3 ft.

Hence, *a Common Multiple of two or more numbers, or quantities, is such a number, or quantity, as contains each of the given numbers, or quantities, a whole number of times.*

THE LEAST COMMON MULTIPLE

In the above, 12 and 24 are common multiples of 3 and 4; and 6 ft. and 12 ft. are common multiples of 2 ft. and 3 ft.

Of these, 12 is the Least Common Multiple of 3 and 4, and 6 ft. is the Least Common Multiple of 2 ft. and 3 ft.

The Least Common Multiple (L.C.M.) of two or more given numbers, or quantities, is the least number, or quantity, which contains each of these numbers, or quantities, a whole number of times.

TO FIND THE L.C.M. OF TWO OR MORE NUMBERS

1. By trial

This method is illustrated in the preceding sections.

2. By use of prime factors

Find the L. C. M. of 24, 35, 90.

Resolving 24, 35, and 90 into their prime factors we have:

$$24=2\times2\times2\times3;$$
$$35=5\times7;$$
$$90=2\times3\times3\times5.$$

To contain 24 the L.C.M. must have the prime factors 2, 2, 2, 3; to contain 35 it must have the factors 5, 7; to contain 90, it must have the factors 2, 3, 3, 5.

Thus to contain 24 the L.C.M. must have 2, 2, 2, 3, as factors.

Thus to contain 24 and 35 the L.C.M. must have 2, 2, 2, 3, 5, 7, as factors.

Thus to contain 24, 35, and 90, there must be only the additional factor 3, since the other factors, 2, 3, 5, are already found.

Thus $2\times2\times2\times3\times5\times7\times3$, that is 2520, is the L.C.M. of 24, 35, and 90.

The L.C.M. may be written $2^3\times3^2\times5\times7$, which shows that the L.C.M. of two or more numbers is the product of all the prime factors of the numbers, each factor being taken the greatest number of times it is found as a factor in any of the given numbers.

3. By the usual device

The following method is often used:

$$\begin{array}{r|rrrr}
2) & 24, & 35, & 90 & (d) \\ \hline
3) & 12, & 35, & 45 & (c) \\ \hline
5) & 4, & 35, & 15 & (b) \\ \hline
 & 4, & 7, & 3 & (a)
\end{array}$$

L.C.M.$=2\times3\times5\times4\times7\times3=2520$.

The divisors used are prime factors common to any two or more numbers. Division is carried on until there is found no factor common to any two numbers.

Since 4, 7, 3, are prime to one another their L.C.M.$=4\times7\times3$, and since the numbers in line (*b*) are either the same as those in line (*a*), or just 5 times those numbers, the L.C.M. of numbers in (*b*) is $4\times7\times3\times5$; so the L.C.M. of numbers in (*c*) is $4\times7\times3\times5\times3$; and the L.C.M. of numbers in (*d*) is $4\times7\times3\times5\times3\times2$, that is, 2520.

This method consists in preserving from all the numbers the factors that will be required in the L.C.M.

4. By using H.C.F.

Sometimes the L.C.M. of numbers not easily resolved into their prime factors is required.

Example: Find the L.C.M. of 5141 and 9991.

First find the H.C.F., which is 97,

Then $5141=97\times53$, and $9991=97\times103$.

Now it is clear that $(97\times53)\times(97\times103)$ is a common multiple, but not the L.C.M., the factor which is common, namely 97, being unnecessarily repeated. Therefore the L.C.M. is

$$(97\times53)\times(97\times1033)\div97 \text{ or } \frac{(97\times53)\times(97\times103)}{97}$$

that is, $\dfrac{\text{the product of the numbers}}{\text{their H.C.F.}}$

CHAPTER XVII

DECIMALS

RELATION TO WHOLE NUMBERS

REVIEW the notation of whole numbers, using for example the number 111. How many times the number represented by the "one" to the right is the number represented by the "one" in the middle position? Ten times.

How many times the number represented by the middle "one" is that represented by the "one" to the left? Ten times.

How does moving a digit one place to the left affect the value of the number represented by the digit? It gives it a value ten times as great.

How does moving a digit one place to the right, say from the third place to the second place, or from the second place to the first place, affect the value of the number represented by the digit? It gives it a value of $\frac{1}{10}$ of what it had.

If this rule were applied further, what would be the value of the number represented by the digit "one" when put one place to the right of the "one" in the units' place? $\frac{1}{10}$ of 1. What would be the value of "one" when put two places to the right of the units' place? $\frac{1}{10}$ of $\frac{1}{10}$ of 1, that is $\frac{1}{100}$. What would be the value of "one" when put three places to the right of the units' place? $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of 1, that is, $\frac{1}{1000}$, etc.

From this it is evident that, if the units' place were marked, there could be other places to the right of it for such fractions as $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., just as there are places to the left of it for 1 ten, 1 hundred, 1 thousand, etc.

The way in which the units' place is marked is to put a *point* immediately after it. Thus

111·111

Once more, give the values of the numbers represented by the different "ones": One hundred, one ten, one thousand, one tenth, one hundredth, one thousandth.

Read all of the number to the left of the point. Now read all of the number to the right of the point. The 1 to the extreme right is 1-thousandth, the next 1, *from its position,* is 10-thousandths, and the next 1 is 100-thousandths, so that to the right of the point there is in all 111-thousandths.

The point is called a *decimal point* and is used to mark the units' place, and thus separate the integral from the fractional part of the number.

The fractions which can be written in the same system of notation as whole numbers are called *Decimals.* What numbers do decimals have for denominators? How can we know what the denominator is in any case?

The pupil should now be asked to write out an extension of the ordinary system of notation showing the names of the places and periods to the left and to the right of the decimal point.

NUMERATION AND NOTATION

Considerable practice should now be given in writing and in reading in different ways, numbers expressed in the decimal system.

The questions given on pages 144 and 145 of the *Ontario Public School Arithmetic* should be thoroughly understood.

Other examples:

1. Read 28·56 as a mixed number; as tenths; as tens; as hundredths.

2. Express ·3, 7·5, ·08 as fractions.

3. Express $3\frac{8}{100}$, $\frac{75}{1000}$, $\frac{8}{10000}$ as decimals.

4. When a fraction may be expressed by giving only its numerator and the position of the numerator, what kind of fraction is it? How is the denominator of the fraction known in this case?

5. How is it that 9, 9·0, and 9·00 have the same value? Write them as fractions and reduce them to their lowest terms.

6. Read 628·7634 and show that the reading properly expresses the number.

7. Show how the insertion of a zero between the point and the 7, in the decimal ·7, changes the value of the decimal, but that a zero after the 7 does not change the value.

8. Name the decimal consisting of two digits which lies nearest in value to ·4382.

ADDITION AND SUBTRACTION OF DECIMALS

Neither of these operations will give any difficulty, if the addends in addition and the minuend and subtrahend in subtraction are written so as to bring the units of the same order in the same vertical columns.

Whatever method of subtraction has been used with integers should be used in subtraction of decimals.

For Exercises, see *Ontario Public School Arithmetic*, pages 147 and 148.

MULTIPLICATION OF DECIMALS

1. Multiply ·6 by 7.

$$\cdot 6 = \frac{6}{10} \therefore \cdot 6 \times 7 = \frac{6}{10} \times 7 = \frac{42}{10} = 4\cdot 2.$$

2. Multiply ·031 by 6.

$$\cdot 031 = \frac{31}{1000} \therefore \cdot 031 \times 6 = \frac{31}{1000} \times 6 = \frac{186}{1000} = \cdot 186.$$

3. Multiply 2·31 by 7.

$$2\cdot 31 = \frac{231}{100} \therefore 2\cdot 31 \times 7 = \frac{231}{100} \times 7 = \frac{1617}{100} = 16\cdot 17.$$

4. Multiply ·35 by ·6.

$$\cdot 35 = \frac{35}{100} \text{ and } \cdot 6 = \frac{6}{10} \therefore \cdot 35 \times \cdot 6 = \frac{35}{100} \times \frac{6}{10} = \frac{210}{1000} = \cdot 210.$$

5. Multiply 3·5 by ·9.

$$3\cdot 5 = \frac{35}{10} \text{ and } \cdot 9 = \frac{9}{10} \therefore 3\cdot 5 \times \cdot 9 = \frac{35}{10} \times \frac{9}{10} = \frac{315}{100} = 3\cdot 15.$$

6. Multiply 2·5 by 3·7.

$$2\cdot 5 = \frac{25}{10} \text{ and } 3\cdot 7 = \frac{37}{10} \therefore 2\cdot 5 \times 3\cdot 7 = \frac{25}{10} \times \frac{37}{10} = \frac{925}{100} = 9\cdot 25.$$

Examine the products and lead the class to observe:

1. That the figures are the same as those got by multiplying the multiplier and multiplicand as if they were whole numbers.

2. That the number of places to the right of the decimal point in the product is equal to the sum of the places to the right of the decimal point in the multiplier and multiplicand.

Example: Multiply 2·37 by 8·6.

The lowest digit in the multiplier is $\frac{6}{10}$ and in the multiplicand is $\frac{7}{100}$. Therefore the lowest result in the product will be a number of thousandths; that is, will stand in the third place to the right of the decimal point.

Exercises in multiplication of decimals are given in the *Ontario Public School Arithmetic,* pages 149 and 150. The following may be added:

1. Multiply the following decimals by 10,
   1·6, ·6, 1·3, 82·7, 1023·6.
2. Multiply the following decimals by 100,
   5·08, 1·37, 805·63.
3. By what must the following be multiplied to convert them into whole numbers?
   ·8, 1·732, ·05, ·0765, ·37856.

DIVISION OF DECIMALS

In the division of decimals, considerable difficulty is experienced from inability to put the decimal point in its proper place in the quotient. This may be overcome by:

1. Making the divisor a whole number.
2. Setting the partial quotients in their proper place *above the corresponding figures in the dividend.* This will at once fix the decimal point for the quotient.

In order that the divisors may be made into whole numbers, it will be necessary to establish the fact that, if divisor and dividend be multiplied by the same number, the quotient is unaltered. This can be done by taking examples in ordinary division:

(a) 8)24(3 (b) 15)90(6 (c) 7)28(4
2)6(3, 3)18(6, 42)168(4, etc.

Illustrations

1. Division by an Integer

Example: Divide 8·46 by 5.

The explanation of the process is exactly the same as in ordinary division.

$$\begin{array}{r} 1{\cdot}692 \\ 5)\overline{8{\cdot}46} \\ \underline{5} \\ 34 \\ \underline{30} \\ 46 \\ \underline{45} \\ 10 \\ 10 \end{array}$$

The quotient is $\frac{1}{5}$ of the dividend; $\frac{1}{5}$ of 8 units is 1 unit.

Set this down above 8 units. This alone determines the decimal point for the quotient.

From the first step of division three units are left over and these read with the 4 tenths = 34 tenths. $\frac{1}{5}$ of 34 tenths = 6 tenths.

Set this down over the 4 tenths and proceed as before.

2. Division by a Decimal

Example: Divide ·00123 by ·04.

$$\begin{array}{r} {\cdot}04){\cdot}00123 \\ \\ {\cdot}03075 \\ 4)\overline{{\cdot}123} \\ \underline{12} \\ 30 \\ \underline{28} \\ 20 \end{array}$$

By multiplying both divisor and dividend by 100 to bring the divisor to an integer, we get ·123÷4, and we then proceed as follows:

Using two figures of the dividend we have 12 hundredths, and $\frac{1}{4}$ of 12 hundredths gives 3 hundredths. Place this above the hundredths and place the decimal point above the decimal point in the dividend and fill in the nought thus, ·03. Then proceed as before.

Example: Divide 1·73546 by 456·7.

$$\begin{array}{r} {\cdot}003 \\ 4567)\underline{17{\cdot}3546} \end{array}$$

By multiplying both terms by 10 to make the divisor an integer, we get 17·3546÷4567.

Use 5 figures of the dividend and read thus, 17354 thousandths. $\frac{1}{4567}$ of 17354 thousandths gives 3 thousandths. Place the 3 thousandths above the 4 thousandths of the dividend and we have for quotient ·003.

An Alternative Method

The method of making the divisor a ***whole number*** before dividing is the one usually adopted for the division of decimals, and is the method perhaps least liable to error in placing the decimal point in the quotient, but it is open to the objection that the question from which the quotient is obtained is not the question actually assigned, but is only an *equivalent* question giving the same answer.

Without interfering with the decimal point in either divisor or dividend, the position of the decimal point in the quotient may be determined from a consideration of the following:

What is the value of \$6÷\$2? of 8 tens ÷ 4 tens? of 9 units ÷ 3 units? of 6 tenths ÷ 3 tenths? of 8 thousandths ÷ 4 thousandths?

In each case what is the *place* value of the digit in the quotient?

If the *divisor* is tenths, what will be the *place* value of the digit which must be included in the dividend in order to get, in the quotient, the digit of the *units' order?*

If, then, the divisor is hundredths, or thousandths, or millionths, what, in each case, will be the place value of the *dividend digit* which will give the *quotient digit* of the units' order?

In the following examples point out how many digits of the dividend have been included in the division when the units' digit in the quotient is obtained, and give reasons for your answer:

49 ÷ 7; 384 ÷ 24; 6·3 ÷ 3; 4·25 ÷ 85; ·42 ÷ 6; ·3178÷73; 34·25÷6·5; 7·189÷4·33; 94·1426 ÷ 3·014; 45·32÷·614; 2·8153÷·00416; 55÷·003; etc.

When the units' digit in the quotient is obtained, the decimal point is, of course, located. Why?

Illustrations

Example: Divide 658·79 by 53.

Here the divisor is *units*. Hence when the *units* of the *dividend* have been divided, the *units'* digit of the quotient is obtained, and the decimal point immediately follows; that is, when 658 is divided, the units' digit in the quotient is reached and the decimal point is placed before the quotient digit which is obtained from including the 7 in the dividend.

Example: Divide 90·3494 by ·26.

Here the divisor is hundredths. Hence the units of the quotient are reached when 90·34 has been divided, and the decimal point is placed before the quotient digit which is obtained from taking the 9 thousandths into the dividend.

Example: Divide 3·7 by ·042.

Here the divisor is thousandths, and so when 3·700 is divided, the units' digit in the quotient is obtained, and the decimal point immediately follows.

TABLES FOR RAPID CALCULATIONS

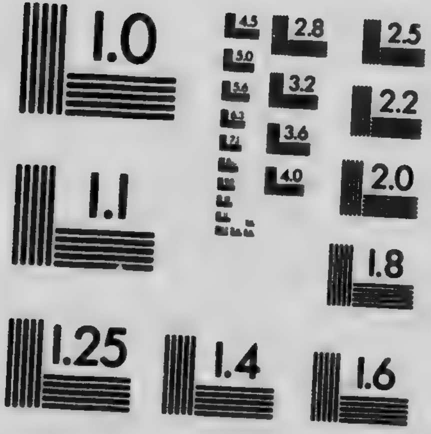
An interesting exercise in division and multiplication of decimals will be furnished by the construction of what we may call "Profit Sharing Tables".

For example, take the following problem:

The Alexandra Cheese Company divides its net receipts among the patrons according to the amount of milk each supplies to the factory during the season. In 1909, the total milk supplied was 1757087 lb. The total receipts from the sale of cheese were $18657·08, and the total expenses were $2387·49. Find the amount which should be paid to each of four patrons who supplied, respectively, 82959 lb., 78040 lb., 72065 lb., and 65457 lb. of milk.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



APPLIED IMAGE Inc
1653 East Main Street
Rochester, New York 14609 USA
(716) 482 – 0300 – Phone
(716) 288 – 5989 – Fax

Here the profits were \$18657·08 — \$2387·49 = \$16269·59.

Hence for 1757087 lb. milk there was paid \$16269·59
“ “ 1 lb. “ “ “ “ $\frac{\$16269\cdot59}{1757087}$
$=\cdot925935$c.

A table may now be constructed to show what was paid for 1, 2, 3, 4, 5, 6, 7, 8, and 9 lb. respectively, thus:

Lb.	Payments	Lb.	Payments	Lb.	Payments
1	·925935c.	4	3·703740c.	7	6·481545c.
2	1·851870c.	5	4·629675c.	8	7·407480c.
3	2·777805c.	6	5·555610c.	9	8·333415c.

To find what was paid to the patron who supplied 82959 lb. it is only necessary to multiply the amount paid on 8 lb. by 10,000, the amount paid on 2 lb. by 1000, the amount paid on 9 lb. by 100, the amount paid on 5 lb. by 10, and the amount paid on 9 lb. by 1, and add the products. The multiplication in each case is effected by shifting the decimal point, and the product can, therefore, be set down at once thus:

The amount paid on 80000 lb. = \$740·748
“ “ “ 2000 “ = 18·518
“ “ “ 900 “ = 8·333
“ “ “ 50 “ = ·462
“ “ “ 9 “ = ·083
“ “ “ 82959 “ = \$768·144

The amounts paid to the other patrons can be determined similarly. Have the pupils find them.

CHAPTER XVIII

COMMERCIAL ARITHMETIC

PERCENTAGE

IF FRACTIONS are properly taught and understood, percentage will present to the pupil nothing new except the terms used. The process involved, as well as the underlying principles, should be so explained and applied in the theory and practice of fractions that the pupils will be led to see that they have, in percentage, but a new name for a fraction with which they are already familiar, and that there is really nothing to warrant its treatment as a separate department of arithmetic except its application to a wide range of problems usually classified under the name of Commercial Arithmetic.

As a preparation for percentage, special attention must be given to the two following operations in fractions:

1. The comparison of one quantity with another so as to be able to express the measure of the one in terms of that of the other; that is, to express the measure of the one when that of the other is taken as the unit of measurement, or, in other words, to express one as a fraction of the other.

2. The reduction of fractions to equivalent fractions of different denominations.

These two are the *fundamental operations* involved in all problems in percentage and its applications, and it remains for the pupil but to select the quantity to be measured and the quantity which is to be used as the unit of measurement. This he can do only by having a clear understanding of the nature of the transaction with which the particular problem under consideration deals.

COMPARISON OF QUANTITIES

The pupil already knows that there is necessarily no fixed unit for the measurement of quantities, and that when the unit is changed the measure likewise changes. For instance, the measure of the quantity, 6 inches, is 6 when 1 inch is the unit of measurement; but it is ½ when 12 inches, or 1 foot, is the unit; and it is ⅙ when 36 inches, or 1 yard, is the unit.

When, therefore, we say that 6 inches is ½ of 1 foot or ⅙ of 1 yard we are really measuring 6 inches by using 12 inches and 36 inches, respectively, as the unit of measurement, or we are, in other words, expressing 6 inches as a fraction of, or in terms of, 12 inches and 36 inches respectively. But 6 inches can, in a similar manner, be measured by or compared with, other lengths besides 12 and 36 inches; such, for example, as 24 inches, 9 inches, 11 inches, 3 inches, 7½ inches, etc., for which there are no corresponding names like one foot and one yard; and when so measured it is necessary to specifically state these lengths.

Illustration

1. 6 inches $= \frac{6}{12}$ of one foot, that is, of 12 inches,
$= \frac{6}{36}$ of one yard, that is, of 36 inches,
$= \frac{6}{24}$ of 24 inches,
$= \frac{6}{9}$ of 9 inches,
$= \frac{6}{11}$ of 11 inches,
$= \frac{6}{3}$ of 3 inches,
$= \frac{6}{7\frac{1}{2}}$ of 7½ inches.

2. 5 pints $= \frac{5}{2}$ of a quart, that is, of 2 pints,
$= \frac{5}{8}$ of a gallon, that is, of 8 pints,
$= \frac{5}{12}$ of 12 pints, or of 6 quarts, or of $1\frac{1}{2}$ gallons,
$= \frac{5}{3}$ of 3 pints,
$= \frac{5}{1\frac{1}{4}}$ of $1\frac{1}{4}$ pints.

3. $10 = \frac{10}{20}$ of 20,
$= \frac{10}{100}$ of 100,
$= \frac{10}{5}$ of 5.

4. So, too, $\frac{1}{2} = \frac{\frac{1}{2}}{2}$ of 2,
$= \frac{\frac{1}{2}}{\frac{1}{3}}$ of $\frac{1}{3}$,
$= \frac{\frac{1}{2}}{100}$ of 100.
etc., etc.

From such illustrations the pupil learns to compare one quantity with another, and to perceive that such comparison can be made only between quantities of the same kind and denomination. So important is this exercise that the teacher will do well to give much and varied practice in it, until the pupil reaches the conclusion that in every case a quantity is measured by, or compared with, some other by finding how often the first quantity contains that other.

GENERAL LAW FOR EQUIVALENT FRACTIONS

The pupil has already discovered that a fraction is changed to an equivalent one by multiplying or dividing each of its terms by the same number. He can now be shown that this is true no matter what that "same number" is, that is, whether it is an integer, a fraction, or a mixed number. For instance, he has learned that

Present — means / fractions with the / denominators always

$\frac{3}{4} = 3 \div 4$; giving this meaning to a fraction let him find the value of such fractions as:

$$(a)\ \frac{2 \times \frac{4}{5}}{3 \times \frac{4}{5}} \qquad (b)\ \frac{3 \times 2\frac{3}{4}}{7 \times 2\frac{3}{4}} \text{ etc.}$$

Here (*a*) is equal to $(\frac{8}{5} \div \frac{12}{5}) = \frac{8}{12} = \frac{2}{3}$ and (*b*) is equal to $(\frac{33}{4} \div \frac{77}{4}) = \frac{33}{77} = \frac{3}{7}$. Similar illustrations will show that the value of the fraction is unchanged when each of its terms is multiplied by the same fraction or the same mixed number. It should be pointed out that when the numerator and denominator of a fraction are multiplied by the "same number" the fraction is really multiplied by *unity*, or its equivalent. For instance, $\frac{\frac{4}{5}}{\frac{4}{5}}$, $\frac{2\frac{3}{4}}{2\frac{3}{4}}$, above, are each equal to 1. Multiplying by 1 does not alter the multiplicand. It is, therefore, possible to change fractions to equivalent ones having any numbers whatever for denominators, as it is evidently not necessary that the new denominators should always be exact multiples of the original denominators.

Illustration

For instance, $\frac{2}{3}$ can be changed to an equivalent fraction having 7 for its denominator,

(*a*) By finding what 3 must be multiplied by to give 7; and then

(*b*) By multiplying the numerator 2 by that "same number".

It is evident that 3 must be multiplied by $\frac{7}{3}$ to give 7 hence, $\frac{2}{3} = \frac{2 \times \frac{7}{3}}{3 \times \frac{7}{3}} = \frac{\frac{14}{3}}{7} = \frac{4\frac{2}{3}}{7}$.

Let the pupil prove this by dividing $4\frac{2}{3}$ by 7.

The pupil should now be given practice in changing fractions to equivalent ones whose denominators are not exact multiples of the denominators of the original fractions. He should be required to prove these results by converting the new fractions into the original ones.

HUNDREDTHS, OR PER CENT.

Finally, he will be asked to change fractions to equivalent ones having 100 for their denominator.

Example: Change $\frac{3}{11}$ to hundredths.

By what must 11 be multiplied to give 100? How do we find this? By what then must 3 be multiplied if the *value* of the fraction is not to be changed?

$$\tfrac{3}{11}=\frac{3\times\frac{100}{11}}{11\times\frac{100}{11}}=\frac{\frac{300}{11}}{100}=\frac{27\frac{3}{11}}{100}.$$

After sufficient drill has been given in exercises of this kind, the pupil may be told that *hundredths* are such convenient fractions to work with that a special name and a special symbol are given to them. The name is *per cent.*, which means hundredths, and the symbol is %. Thus $\frac{10}{100}$ is sometimes written 10 per cent., and sometimes 10%. In what other way can it be written? ·10. Have the pupil find out wherein lies the convenience of hundredths.

Exercises will now follow in expressing fractions, both vulgar and decimal, in per cent., and per cent. in fractions; and the pupil should memorize the fraction equivalents of some of the most commonly used per cents.

PERCENTAGE PROBLEMS

Problems in percentage group themselves into three general classes and, no matter how complicated the problem, its solution may be found by applying the principles involved in the solutions required in these three classes taken singly or in combination. Here, as elsewhere, the teacher should impress the fact that all problems, however difficult, can be resolved into a series of very simple ones that require for their solution but one step at a time.

Standard, or Unit Quantity	Rate %	Measured Quantity

The classes are:

1. Given a standard quantity, and a rate per cent., to find the quantity measured.

2. Given a standard quantity, and the quantity measured, to find the rate per cent.

3. Given the quantity measured and the rate per cent., to find the standard quantity.

Examples

1. As an example of the first class, take the problem:

A man is able to save $16\frac{2}{3}\%$ of his income. How much will he save in five years out of an annual income of \$1200?

Here the standard quantity is his income.

The quantity measured is his savings, and the rate per cent. is $16\frac{2}{3}$.

$$\begin{aligned}\text{Now, savings} &= \left(16\tfrac{2}{3}\% \text{ or } \frac{16\frac{2}{3}}{100} \text{ or } \tfrac{1}{6}\right) \text{ of income,}\\ &= \tfrac{1}{6} \text{ of } \$6000,\\ &= \$1000.\end{aligned}$$

2. As an example of the second class take the problem:

In 1910 a town had a population of 2500. In 1911 the population was 2820. What per cent. is the increase of the population in 1911?

What is the standard quantity? The population in 1910.

What is the measured quantity? The increase of population.

What is the increase in population? 320.

With what is this to be compared? With 2500.

What fraction is 320 of 2500? $\frac{320}{2500}$ of 2500.

What is the fraction when reduced to *hundredths?*

$\frac{\frac{320}{25}}{100}$ or $\frac{12\frac{4}{5}}{100}$.

What then is the rate per cent.? $12\frac{4}{5}\%$.

3. As an example of the third class take the problem:

A merchant failing in business is able to pay but 35% of his debts. What does he owe a bank to which he is able to pay $175?

What quantities are here compared? The man's payments and his debts.

What is the relation between these? His payments are 35% or $\frac{7}{20}$ of his debts.

What sum is mentioned in the problem? $175.

Is this a payment or a debt? A payment.

Is the sum to be found a payment or a debt? A debt.

How then is the debt required connected with the payment made? $\frac{7}{20}$ of the debt to the bank = $175.

Therefore the debt to the bank = $\frac{20}{7}$ of $175, or $500.

DEVICES

To express the per cent., decimal fractions may be used as well as vulgar fractions, and they will sometimes be found more convenient.

For instance, in the first example, if the man had saved 12% of his income, his total savings would be $\frac{12}{100}$ of \$6000, that is, \$6000 × ·12=\$720.

In the second example $\frac{320}{2500} = \cdot 128 = 12\cdot 8\%$.

In the third example ·35 times the debt = \$175;

$$\text{Therefore the debt} = \frac{\$175}{\cdot 35} = \$500.$$

A fraction may also be converted into per cent. by considering the fraction as a fraction of the whole of some number, or quantity, that is, as a fraction of 100% of that number, or quantity. For instance, in the second example above, the increase was found to be $\frac{320}{2500}$ or $\frac{32}{250}$ of the 1910 population. Therefore it was $\frac{32}{250}$ of 100% of that population. That is, it was $(\frac{3200}{250}$ or $12\frac{4}{5})\%$ of the population.

To make the pupils realize the practical importance of percentage the teacher should set before them, and get them to construct for themselves, problems which are constantly arising in actual experiences, such as calculating the per cent. which the average daily attendance is of the total school enrolment; the per cent. that the total marks obtained by the pupil at an examination are of the total marks obtainable; the per cent. of the pupils in a class having the correct answer to any particular question; the per cent. of the boys that are correct; the per cent. of the girls; and many other problems which will be suggested in connection with school experiments and investigations. From such exercises the pupil will see how percentage assists us in making comparisons.

With a thorough understanding of percentage, of its nature, underlying principles, and fundamental operations, the pupil may proceed to its application in various business and commercial transactions. In doing so his *one* difficulty is likely to result from his inability, through lack of actual business experience, to properly grasp the nature of the transactions involved and the meaning of the special terms employed in the problems with which he may have to deal. Consequently, it will now become the teacher's one care to remove that inability by supplying what the pupil lacks.

PROFIT AND LOSS

There is nothing characteristic enough about problems in Profit and Loss to warrant special treatment of them. The pupils must be told that the quantity with which the loss or gain is compared or in terms of which it is expressed (that is, by which it is measured) is the *cost*, unless specific statement is made, in the problem, to the contrary. The selling price, it is evident, can also be expressed in terms of the cost. With this information and with a clear understanding of the nature of the transactions involved, the pupil should require little further assistance from the teacher.

There are certain special terms used in these problems, the meanings of which the pupils must know. The terms are: *Cost price, selling price, marked price, gross price, list price, net price, profit, loss,* and *discount.*

To make clear the nature of a profit and loss transaction as well as the meaning of the terms used, the teacher will find it exceedingly helpful to have a practical illustration given to the class as follows:

Select a pupil to act the part of a merchant buying goods and selling them. Let a book represent the goods. The first thing which the merchant will do is to write on the book the amount it cost him, that is the *cost price.* This he usually does in *private* marks or letters instead of the ordinary figures. For instance, he may take the letters of the words "r-e-d c-u-s-h-i-o-n", and instead of writing the figure 1 he writes *r*; for 2 he writes *e*; *d* stands for 3; *c* for 4; *n* for 0, etc. If the cost is $1.05 the merchant will write on the book *r n u.* He then decides to sell it at a *profit* of say 33⅓%. The gain or *profit* would then be 35c. On the book will then be written, usually in ordinary figures, $1.40, which is called the *marked price,* or *list price,* and is intended also to be the *selling price.* A favoured customer, however, buys the book and the merchant "throws off", that is, gives a *discount* of 10% from the *marked price.* This discount amounts to 14c. and the book is then sold for $1.26, which is the *net selling price.*

In carrying out the illustration, the boy selected will, assisted by the other pupils, proceed as directed in the above explanations, suggest the cost, and the rates of prof and discount, and make all the necessary calculations.

This exemplification of the transaction should be followed by oral exercises to establish the main underlying principles, after which written exercises may be given.

PROBLEMS

Cost	Rate %	Gain or Loss

The problems group themselves into three general classes:

1. Given the cost and the rate, to find the gain, loss, or selling price.

2. Given any two of the gain or loss, cost, and selling price, to find the rate.

3. Given the gain or loss, or given the selling price, and given the rate, to find the cost.

When the problem can be immediately placed in one or other of the above classes, its solution will be readily obtained, but the classification is sometimes complicated, and the problem will then require more careful analysis.

Examples

1. A merchant sells carpet at 75c. a yard and gains 20%. What will be his gain per cent. if he raises the price to 85c. a yard?

This problem may be divided into simpler problems, one of which will belong to the third general class, and the other to the second class.

$$\begin{aligned}
\text{Cost} + \tfrac{1}{5} \text{ of cost} &= 75\text{c.},\\
\tfrac{6}{5} \text{ of cost} &= 75\text{c.},\\
\text{cost} &= \tfrac{5}{6} \text{ of } 75\text{c., or } 62\tfrac{1}{2}\text{c.},\\
\therefore \text{second gain} &= 22\tfrac{1}{2}\text{c.},\\
&= \frac{22\frac{1}{2}}{62\frac{1}{2}} \text{ of cost},\\
&= (\tfrac{45}{125} \text{ or } \tfrac{36}{100}) \text{ of cost},\\
&= 36\% \text{ of cost};
\end{aligned}$$

Or,

$$\begin{aligned}
&75\text{c.} = 120\% \text{ of the cost},\\
&85\text{c.} = \tfrac{85}{75} \text{ of } 120\% \text{ of the cost} = 136\% \text{ of the cost},\\
&\text{gain} = (136-100)\% \text{ of the cost} = 36\% \text{ of the cost}.
\end{aligned}$$

2. At what price must goods costing $40 be marked (or listed) so that a discount of 16⅔% may be allowed and a profit of 20% still realized?

This problem can be divided into two separate problems, one of which is in the first general class, and the other is a modification of the third general class.

Have the pupils state and solve these problems separately, and compare their solution with the following:

$16\frac{2}{3}\% = \frac{1}{6}$; $20\% = \frac{1}{5}$.
$\frac{5}{6}$ of the marked or list price $= \frac{6}{5}$ of the cost (why?).
The marked or list price $= \frac{6}{5}$ of $\frac{6}{5}$ of the cost,
The marked or list price $= \frac{36}{25}$ of the cost,
The marked or list price $= \frac{36}{25}$ of \$40,
The marked or list price $=$ \$57.60.

3. A railway messenger boy buys 10 doz. oranges at the rate of 4 for 10 cents and sells them at the rate of 3 for 10 cents. Find his gain per cent. if 10% of the oranges are unsaleable.

This problem belongs to the second general class. Have the pupils solve it and compare with the following:

Cost $= \frac{10}{4}$c. or $2\frac{1}{2}$c. an orange.
Selling price $= \frac{9}{10}$ of $\frac{10}{3}$c. or 3c. an orange. (Why?)
Gain $= \frac{1}{2}$c. an orange.
$= \frac{1}{5}$ of cost (why?),
$= 20\%$ of cost.

COMMISSION

Transactions in Commission may be exemplified in the class-room in a manner similar to that detailed under Profit and Loss. The pupils may suggest commodities to be dealt in, prices to be received or paid therefor, as well as the *rate* of commission to be allowed, and they will make the necessary calculations demanded by the problems which arise in connection with the transactions illustrated.

The pupils must be told that commission is expressed in terms of the *total amount* for which the *agent* sold in the case of sales, and in terms of the amount which the *agent* paid or collected in case of purchases or collections. In other words, commission is always expressed in terms of the amount involved in the transaction.

The special terms, the meaning of which must be clearly understood, are: *Agent, broker, commission merchant, principal, consignor, investment, proceeds, commission,* and *net proceeds.*

PROBLEMS

Amount of Sale or Amount of Purchase	Rate %	Commission

The problems will again group themselves into three general classes:

1. Given the total amount for which the agent sold or the amount for which the agent bought, and the rate, to find the commission.

2. Given the commission and the amount for which the agent sold or the amount for which he bought, to find the rate.

3. Given the commission and the rate, to find the amount for which the agent sold or the amount for which he bought.

The principles underlying these three classes should be illustrated and explained by oral problems, and followed by those requiring written work.

Of the two transactions, buying and selling, pupils appear to experience more difficulty in understanding the former. But this is due chiefly to their losing sight of the fact that the commission for buying is expressed in terms of, or measured by, the *amount paid by the agent* for the goods he buys.

Examples

1. I send a real estate agent in Calgary $5,040 with which to purchase land at $15 an acre, and pay his commission therefor at 5%.

How many acres can he buy and what will be his commission?

First Solution (by question and answer)

$5040 is made up of what two amounts? The amount paid for the land by the agent and the agent's commission.

Make that statement in the form of an equation.

(*a*) Amount paid for land by agent + agent's commission = $5040.

What is the agent's commission? 5% of the amount paid by him for the land.

Then re-write the first statement expressing the agent's commission in terms of the amount paid for the land.

(*b*) Amount paid for land + $\frac{5}{100}$ of amount paid for land = $5040.

Combine these two amounts and express the whole in terms of the amount paid for the land.

(*c*) $\frac{105}{100}$ of amount paid for land = $5040.

What then is the amount paid for the land? $\frac{100}{105}$ of $5040, or $4800. How many acres will this amount buy? $4800 ÷ $15, or 320.

What now is the agent's commission? $5040—$4800, or $240.

Look again at statement (*c*) and give another way by which the commission could be found.

(*d*) $\frac{105}{100}$ of the amount paid for the land $= \$5040$,

$\frac{5}{100}$ of the amount paid for the land $= \frac{5}{105}$ of $\$5040$,

$=$ commission.

Second Solution

This problem might have been worked more directly, thus:

Cost of land to agent $= \$15$ an acre,

Commission $= \frac{5}{100}$ of $\$15$ an acre,

$= \$.75$ an acre,

$\therefore$ total cost $= \$15.75$ an acre,

No. of acres bought $= \frac{504000}{1575}$.

What then is the commission?

The first solution is the more general of the two, and when properly understood may be applied to a wider range of problems than can the second solution.

2. An agent received $3060 with which to buy goods and pay therefor a commission of 2%. What was the agent's commission?

Cost of the goods + agent's commission $= \$3060$,

Cost of the goods + $\frac{2}{100}$ of cost of goods $= \$3060$,

$\therefore$ $\frac{102}{100}$ of cost of goods $= \$3060$,

Hence commission, or $\frac{2}{100}$ of cost of goods $= \frac{2}{102}$ of $\$3060$,

$= \$60$.

3. An agent sold some goods on a commission of 5% and invested the proceeds in other goods on a commission of 2%. His total commission was $140. Find the amount for which the first goods were sold.

1st commission $= \frac{5}{100}$ of amount of sales,
$\therefore$ Amount of purchases + 2nd commission
$= \frac{95}{100}$ of amount of sales,
Amount of purchases $+ \frac{2}{100}$ of amount of purchases
$= \frac{95}{100}$ of amount of sales,
$\therefore \frac{102}{100}$ of amount of purchases $= \frac{95}{100}$ of amount of sales.
Hence 2nd commission or $\frac{2}{100}$ of amount of purchases
$= \frac{2}{102}$ of $\frac{95}{100}$ of amount of sales,
$= \frac{95}{5100}$ of amount of sales.
Total commission $= (\frac{5}{100} + \frac{95}{5100})$ of amount of sales,
$= \frac{7}{102}$ of amount of sales.
Hence $\frac{7}{102}$ of amount of sales = $140.
Amount of sales $= \frac{102}{7}$ of $140 = $2040.

TAXES

The teacher will introduce this topic by a talk with the pupils on the need of taxes, the various purposes for which they are used, the authority by which they are levied and collected, how the amount of the taxes required is arrived at, how this amount is proportioned among the ratepayers, the duties of the assessor and of the collector, the difference between the value and the assessment of property, the form and use of an assessment notice, of a tax bill, and of an assessor's and collector's roll. Copies of these forms should be procured if possible. The illustrations used in this talk should be taken from the locality in which the school is situated, so that the pupils may be able to supply most of the information in response to the teacher's questions.

The meaning of the following terms must be made clear: *Tax, income-tax, poll-tax, total and individual assessment, rateable property, assessable property, exemption, ratepayer, rate, levy, assessor, collector.*

The pupil must also be informed that the tax is usually levied in *mills*—a certain number of mills being collected for every dollar for which a ratepayer is *assessed*. A man's property may or may not be assessed for its *actual value*, but in any case it is on the *assessment* that the tax is levied.

PROBLEMS

Assessment	Rate	Tax

The problems in taxes may also be grouped into three general classes:

1. Given the assessment and the tax to find the rate.
2. Given the assessment and the rate to find the tax.
3. Given the tax and the rate to find the assessment.

Their solutions demand little more than a knowledge of the application of multiplication and division.

Practice should be given in making calculations by using a tax table. (See *Ontario Public School Arithmetic*, page 170.)

DUTIES, CUSTOMS, AND EXCISE

These are taxes, and the teacher will be expected to point out wherein they differ from those with which the pupils are already familiar. This he will do by giving the information which will enable the pupils to answer such questions as: For what purpose are these taxes used? On what are they levied? By what authority are they levied and collected? Where and by whom are they collected? Who pays them and when are they paid? What was the total amount of these duties in Canada for the past year?

What is the rate of duty paid on some articles found in general use in the locality in which the pupils live, etc.?

A copy of the Canadian tariff would prove useful, as would also clippings from the newspapers in which are published, from time to time, the customs returns for the Dominion of Canada as well as for districts in which custom-houses are situated.

The meanings of the following terms must be made clear: *Customs, duties, revenue, exports, imports, excise, custom-house, port of entry, invoice, tariff, preferential tariff, rate of duty.*

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KINDS OF DUTIES

The pupil must now know that there are two kinds of duties: (1) Specific, and (2) Ad Valorem.

The *specific duty* is a specified amount levied on stated quantities of the goods taxed, such as: 5 cents on a pound, 20 cents on a gallon, 50 cents on each article, etc.

Problems involving specific duties are, therefore, problems in multiplication.

The *ad valorem duty* is expressed in terms of the *invoiced* value of the goods, that is, it is usually stated as a certain per cent. of that value.

Keeping this in mind, the pupils will readily see that the problems involving ad valorem duty can be grouped again into three general classes and can be solved by the application of principles already established in former exercises.

It will be a profitable exercise to have the pupils make out invoices for themselves for goods ordered from a foreign country, and calculate the duty which would be paid thereon.

SIMPLE INTEREST

GENERAL METHOD

A little consideration will show that in the preceding sections there has been employed one general method which has only to be slightly modified to make it applicable to the particular section which may be under discussion. This method may now be summarized as follows:

1. Make clear the nature of the transactions which give rise to the problems to be solved.

2. Make clear the meanings of the special terms peculiar to the various classes of transactions.

3. Make clear that all problems, in so far as they are percentage problems, may be grouped into the same three general classes, and are, therefore, solved by the application of the same general principles underlying this classification.

It will be noted further that the division of commercial arithmetic into sections is caused by giving a particular name to a special percentage. Sometimes it is called *gain* or *loss*, sometimes *commission*, sometimes *discount*, sometimes *duty*, etc. So, too, there is in each section a particular quantity with which the percentage is compared or by which it is measured. Sometimes it is the *cost*, sometimes the *value of the sales*, sometimes the *value of the purchases*, sometimes the *invoiced value*, etc. And while the names given to the percentages classify the problems in commercial arithmetic, the selection of the particular quantity to be used in any case as the basis of comparison unifies their solutions.

It will no longer be necessary to give the explicit details of the general method, but it will be sufficient to point out the modifications which may be demanded for the problems in the several sections yet to be discussed.

INTEREST EXPLAINED

Interest will be defined as money paid for the use of money, and corresponds to the rent of a farm or house, or the hire of a carriage. A farm is rented for a year, a house for a month, a carriage for an hour, etc. At the end of these periods the owner of the farm, house, or carriage gets back his property and in addition gets the sum agreed upon as rent. So, in a like manner, money is rented, or borrowed, usually for one year, and is then returned to the owner with the additional rent called *interest.* The rent charged per annum is generally a stated per cent. of the sum borrowed. Here, it will be learned that the new name for the percentage is *interest,* and that the quantity, in terms of which the interest is expressed, is the *sum borrowed,* also called the principal sum, or briefly, the *principal.* It will be seen also that there is now introduced into the problems an entirely new element, that of *time,* but that so long as the time is constant the problems may be grouped into the three general classes. It is apparent, therefore, that every interest problem may really be broken up into two minor problems, one a percentage problem, and the other a problem arising out of the introduction of the time element, and this would naturally suggest that practice should be given on these two problems, first separately and then in combination.

SERIES OF PROBLEMS INVOLVED

A. Time Constant

The first exercise in interest will, therefore, consist of problems for oral and written solution, in which the time will be kept constant, that is, kept at one interest period, usually one year. The problems will then be grouped as follows:

(*a*) Given the principal and rate, find the interest.

(*b*) Given any two of the amount, principal, and interest, find the rate.

(*c*) Given the amount, or the interest, and the rate, find the principal.

Between the solution of these problems and of those on Profit and Loss the pupil will not fail to mark the similarity.

B. Rate and Principal Constant

The second exercise will consist of problems into which the element of time is introduced, but the rate and principal remain constant. These may be grouped thus:

(*a*) Given the interest for one year, find the interest for two years, for 1½ years, for 4 months, for 65 days, etc.

(*b*) Given the interest for 1½ years, or 4 months, or 96 days, etc., find the interest for one year.

(*c*) Given the interest for one year, find the time in years, or months, or days, in which a certain other sum would be the interest.

C. Combination of A and B

The third exercise will consist of problems combining the principles involved in the two preceding exercises.

1. Combining (*a*) of A with (*a*) of B, we get a problem in which the principal, rate, and time, are given, and the *interest,* or *amount,* is to be found.

2. Combining (*b*) of A with (*b*) of B, we get a problem in which the principal, or the amount and the interest, and time, are given, and the *rate* is to be found.

3. Combining (*a*) of A with (*c*) of B, we get a problem in which the principal, rate, and interest, or amount, are given, and the *time* is to be found.

4. Combining (*c*) of A with (*b*) of B, we get a problem in which the amount, or interest, the rate, and the time, are given, and the *principal* is to be found.

D. Other Types

A fourth exercise containing problems of the following type will be found useful:

1. Given the fraction of the principal which represents the interest, or the amount, for one year, find the fraction of the principal which would represent the interest, or the amount, for a specified number of years, months, or days.

2. Given the fraction of the principal which represents the interest, or the amount, for any specified time, find the fraction of the principal which would represent the interest for one year; then find the rate.

3. Given the fraction of the principal which represents the interest for one year, find the time in which some other fraction of the principal would represent the interest, or the amount.

ADDITIONAL POINTS TO BE NOTED

It may be added that, in actual business, it is seldom that *simple* interest is calculated for periods extending beyond one year.

While studying interest the pupil should be taught the use and form of commercial papers, such as: *Promissory notes; demand notes; receipts; cheques; deposit slips; bank books showing deposits, withdrawals, balances, and interest.* He should be taught how to calculate the interest on a note, the interest on a savings' account according to the daily balance plan and the minimum monthly balance plan, and the interest on a note on which partial payments have been made and endorsed thereon at irregular periods.

In computing the monthly interest due on savings' accounts, banks reckon the time in *days.* For instance, if the smallest balance for February was $260 and the rate of interest 3%, the February interest would be $\frac{28}{365}$ of $\frac{3}{100}$ of $260. If the smallest balance for March was $826, the interest for March at 3% would be $\frac{31}{365}$ of $\frac{3}{100}$ of $826.

(See *Ontario Public School Arithmetic,* Exercise 82, Questions 16 and 18; also Exercise 99, Questions 66, 67, 110, 137.)

PROBLEM SOLVED

A savings' bank which pays interest at the rate of 4% per annum makes up its interest on 30th June. If the interest on any account is equal to the difference between the interest up to 30th June on all sums withdrawn and the interest on all sums deposited, find the amount of Henry Rust's account on 30th June, if on 10th January he deposits $200, on 15th February withdraws $150, on 25th February deposits $250, on 4th April deposits $400, on 8th May withdraws $200, on 30th June deposits his interest.

(*a*) Deposits

$$171 \text{ days' interest on } \$200 = \$200 \times \cdot 04 \times \tfrac{171}{365} = \frac{\$1368}{365},$$

$$125 \text{ days' interest on } \$250 = \$250 \times \cdot 04 \times \tfrac{125}{365} = \frac{\$1250}{365},$$

$$87 \text{ days' interest on } \underline{\$400} = \$400 \times \cdot 04 \times \tfrac{87}{365} = \frac{\$1392}{365},$$

Total deposits = $850.

$$\text{Total interest on deposits} = \frac{\$4010}{365},$$

(*b*) Withdrawals

$$135 \text{ days' interest on } \$150 = \$150 \times \cdot 04 \times \tfrac{135}{365} = \frac{\$810}{365},$$

$$53 \text{ days' interest on } \underline{\$200} = \$200 \times \cdot 04 \times \tfrac{53}{365} = \frac{\$424}{365},$$

Total withdrawals = $350.

$$\text{Total interest on withdrawals} = \frac{\$1234}{365},$$

$$\begin{aligned} \text{Balance on deposit} &= \$850 - \$350. \\ &= \$500. \end{aligned}$$

$$\begin{aligned} \text{Balance interest} &= \frac{\$4010 - \$1234}{365}, \\ &= \frac{\$2776}{365} = \$7 \cdot 60. \end{aligned}$$

$$\text{Total balance} = \$500 + \$7 \cdot 60 = \$507 \cdot 60.$$

TRADE DISCOUNT

In connection with the transactions involving Trade Discount the information will be given that most goods which are offered for sale in the ordinary stores pass from the producer or manufacturer to the wholesale merchant, and then to the retail merchant, who sells them to those who finally use the goods. The difference between the business of a wholesale merchant and that of a retail merchant might be emphasized, as well as the part performed by the commercial traveller in the sale of goods. It will be noted, of course, that some manufacturers or producers sell directly to the retail merchants.

The manufacturer and wholesale merchant usually issue a *catalogue*, or *list*, of goods which they have for sale, and in this list the goods are *marked*, or quoted, at prices which are as a general rule equal to, if not actually higher, than those for which the consumer buys the goods from the retail merchant.

This catalogue is sent to the *trade*, that is, to the merchants who trade in these goods, and these merchants are notified that to the *trade* there will be allowed a discount of a certain percentage of the *list price*. To induce buyers to take a certain class of goods, to take larger quantities, or to make prompter payments, the price of the goods may be further reduced by offering a second discount and sometimes a third or a fourth. Each discount is a percentage of what *remains after the previous discounts have been deducted.*

In these transactions the name for the percentage of the *list*, or *marked*, price is *trade*, or *mercantile*, or *commercial discount*. After all discounts are allowed, the part of the price left is called the *net* price.

When the goods are forwarded to the retail merchant, the wholesale merchant sends him an invoice giving the quantities of the goods purchased, their list prices, the discounts allowed, and the net amount to be paid.

The pupils should be taught the form of such invoices and given practice in making them up. For this purpose, trade catalogues may be easily secured.

In Trade Discount there are the usual three general classes of problems. State them.

List Price	Rate	Discount

Examples

1. A discount of 20, 10, and 10 off is equivalent to what single discount?

Solution

The net price $= \frac{9}{10}$ of $\frac{9}{10}$ of 80% of list price,
$= (\frac{648}{10}$ or $64{\cdot}8)\%$ of list price,
Therefore actual discount $= (100-64{\cdot}8)\%$ of list price,
$= 35{\cdot}2\%$.

2. At what per cent. above cost must a merchant mark goods so that he can give a discount of 25 and 10 off and still make a profit of 35%?

Solution

The net price $= \frac{9}{10}$ of $\frac{3}{4}$ of the marked price,
$= \frac{27}{40}$ of the marked price.
But net price also $=$ 135% of the cost price.
Hence $\frac{27}{40}$ of marked price $=$ 135% of the cost price,
Therefore the marked price $= \frac{40}{27}$ of 135% of the price,
$= 200\%$.

That is, the goods must be marked at a profit of 100%.

INSURANCE

Beyond the explanation of the nature of the transaction involved and of the technical terms used, *insur-*

ance demands no special treatment. The new name for the percentage is the *premium,* or the cost of insurance; and the quantity in terms of which the percentage is expressed is the *amount of the risk,* that is, the amount which, in accordance with the conditions of the *policy,* will be paid to the *insured* by the *insurer* as a compensation for the loss of the property insured. An insurance policy would add interest to the lesson.

PROBLEMS

Risk	Rate	Premium

For what must a building worth $3000 be insured at $1\frac{1}{4}\%$ in order to cover, in case of loss, the value of the building and the premium paid?

Solution

Here the risk = value of the building + the premium,
That is, the risk = value of the building + $\frac{5}{400}$ of the risk,
Hence $\frac{395}{400}$ of the risk = value of the building,
$\frac{395}{400}$ of the risk = $3000,
$\therefore$ the risk = $\frac{400}{395}$ of $3000.

BANK DISCOUNT

NOTE WITHOUT INTEREST

Perhaps in no other department of commercial arithmetic will the teacher find a practical illustration more helpful than in connection with Bank Discount. Have the pupils draw a note, first without interest, for a specified sum and for a stated time. Then suppose the note *cashed,* that is, converted into cash, at an imaginary school bank over which a selected pupil may preside as banker, and at which the rate of discount is, say, 7%. At first let the note be cashed on the day on which it was drawn. Then suppose it cashed several days after it was drawn. The pupils will see the difference between the proceeds from the two transactions and will understand the reason.

NOTE WITH INTEREST

Now have the pupils draw a second note, this time bearing interest at 6%, but for the same sum and time as before. Once more suppose it converted into cash, or sold to the bank, first on the day it was drawn, and then several days after it was drawn, and let the bank's rate of discount be again 7%. The pupils will see that the proceeds are different in these two transactions and also differ from those obtained in either of the former transactions.

In the course of this illustration the teacher will explain how money is obtained from banks by means of notes, and bring out the fact that the transaction involves two problems, first, one in interest, and secondly, one in discount.

INTEREST PROBLEM

In the interest problem we learn: The date on which the note was drawn; the interest period, that is, the time for which the note is given; the dates on which the note becomes (*a*) *nominally,* and (*b*) *legally* due, the latter date being *three days,* known as *days of grace,* later than the former; the day of maturity; the rate of interest; the face value of the note; the value of the note when it matures, that is, on the day of maturity.

With perhaps the one exception, that of days of grace, all these points in connection with a note fall under transactions in simple interest, and require only reviewing in order to emphasize their special bearing on Bank Discount.

BANK DISCOUNT AND TRADE DISCOUNT

It is in the second problem that the new features of discount enter. It must be noted that Bank Discount is not *interest* but a *discount,* differing from ordinary discounts only in the fact that its amount is conditioned on a time element just as the amount of interest is also dependent on *time.* Here, then, the new name for the percentage is *Bank Discount,* and the quantity on which that percentage is expressed is the *value of the note on the day of maturity.* This percentage is one year's discount, and the amount of it for any portion of a year can then be easily determined. The time for which the discount is to be found is called the *term of discount,* and is the time between the day on which the bank cashes the note and the day on which the note becomes legally due. This period is sometimes called the *unexpired time* of the note.

DISCOUNT PROBLEM

Hence in our second problem, that is, the problem directly involving Bank Discount, we learn: The day of discount, that is, the day the bank cashes or buys the note; the term of discount; the rate of discount; the value of the note at maturity; one year's discount off that value; the discount for the term of discount; and, finally, the proceeds, that is, the amount of cash the bank pays for the note, which proceeds will be equal to the value of the note at maturity, less the discount.

USEFUL EXERCISES

Each item of information secured from both the interest and discount phases of the problems is so important that it is suggested that practice should be given in finding each item separately, before attempting the whole process involved in finding the proceeds obtained from discounting a note.

For instance, it would be well to write on the blackboard a set of, say, six notes, and have the pupils give for each of these, first, their dates; then, their interest periods; then, their dates of maturity; then, their face values; etc.

To assist in making rapid calculations, banks are supplied with numerous tables. The following, by which the number of days between any two specified dates is found, may prove helpful to the teacher.

TABLE OF DAYS INTERVENING BETWEEN DATES

—	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	333	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

METHOD OF USING THE TABLE

The number of days from any day in one month to the *same* day of another month is found by starting at the name of the first (in the left-hand column) and following across to the column headed with the name of the second.

Suppose it is required to find the number of days from March 3 to August 10. From the table we find that it is 153 days from March 3 to August 3; adding 7 days, we find the required time to be 160 days.

Should February 29 of a leap year intervene between dates, add one day.

CHAPTER XIX

SQUARE ROOT

FUNDAMENTAL FACTS

Since $1 \times 1 = 1$; $2 \times 2 = 4$; $3 \times 3 = 9$; $4 \times 4 = 16$; $5 \times 5 = 25$; $6 \times 6 = 36$; $7 \times 7 = 49$; $8 \times 8 = 64$; $9 \times 9 = 81$; $10 \times 10 = 100$; 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100 are said to be the *squares* of 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 respectively.

Also 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are said to be the *square roots* of 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100 respectively.

That 81 is the square of 9 is sometimes expressed thus: 81 = the square of 9; and sometimes, $81 = 9^2$; where the 2 shows that the 9 is to be used as a factor *twice.*

That 7 is the square root of 49 is expressed thus: $7 = \sqrt[2]{49}$ or $7 = \sqrt{49}$ with the 2 omitted.

SQUARE ROOT FOUND BY FACTORING

A square number like 49 is seen to be the product of two equal factors; and the square root of 49 is seen to be *one* of these two *equal* factors.

If, then, we can resolve a number into its two equal factors, one of these factors will be its square root.

Example: Find the square root of 1764. The work may be shown thus:

$$\begin{array}{r}
2\,)\underline{1764} \\
2\,)\underline{882} \\
3\,)\underline{441} \\
7\,)\underline{147} \\
7\,)\underline{21} \\
3
\end{array}$$

$$\begin{aligned}
\text{Thus } 1764 &= 2\times2\times3\times7\times7\times3, \\
&= (2\times3\times7)\times(2\times3\times7), \\
&= 42\times42,
\end{aligned}$$

$\therefore$ the square root of 1764 is 42.

The square root of a number is one of the two equal factors of that number.

For examples, see *Ontario Public School Arithmetic.*

NUMBER OF DIGITS IN SQUARE ROOT

(1) $1^2=1$, and $9^2=81$. Therefore, (*a*) the square of a number of *one* digit is a number of *one* or *two* digits; and (*b*) the square root of a number of *one* or *two* digits is a number of *one* digit.

(2) $10^2=100$ and $99^2=9801$. Therefore, (*a*) the square of a number of *two* digits is a number of *three* or *four* digits; and (*b*) the square root of a number of *three* or *four* digits is a number of *two* digits.

(3) $100^2=10000$ and $999^2=998001$. Therefore, (*a*) the square of a number of *three* digits is a number of *five* or *six* digits; and (*b*) the square root of a number of *five* or *six* digits is a number of *three* digits.

Hence, to find how many digits there are in the integral part of the square root of a number, begin at the decimal point and mark off the digits into groups of two digits each, and there will be as many figures in the integral part of the square root as there are groups of two figures. If there is but one digit in the last group, this is to count as a group. Thus in the integral part of the square root of 1728 there will be two digits. In the integral part of the root of 14641 there will be three digits, because there are three groups or what is counted as such.

USUAL PROCESS

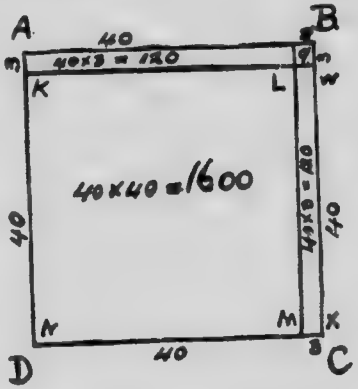
To find the square root of numbers not easily resolved into two equal factors.

Example: Find the square root of 1849.

1. By dividing into groups of two figures each we see there will be two digits in the integral part of the square root, that is, there will be tens and units.

2. By inspection the number is greater than (4 tens)^2 or 40^2 and less than 50^2 and is, therefore, 40 + some other digit.

Represent the number 1849 by a square ABCD. Mark off a part NKLM, 40 by 40. This will take up 1600 and we shall have remaining 1849 — 1600=249. The two strips on the sides will each be 40 long and therefore together 80 long. The problem is to find their width, which added to 40 will give the number which is the square root.



It will be seen that these strips added to the square already found will not make the large square, but need to be supplemented by a small square whose side is the width of the strips. If we call this width a, we have, to make up the large square, a strip $(2\times40+a)$ long and a wide. But this amounts to 249.

From this, since the length of a strip is approximately 80, we see that *a*, or the width, must be 3; and the square root of the number 1849, or the side of the square, is 40+3=43.

The work may be set down as follows:

$$\begin{array}{rl} & 1849\ (\ 40+3\ ) \\ 40\times 40=40^2= & \underline{1600}\ \text{used up in square NKLM} \\ & 249\ \text{remain for strips and small square.} \end{array}$$

$$\left\{\begin{array}{l}(2\times 40)+3,\\ \text{length of strips}\\ \text{and small}\\ \text{square}\end{array}\right\}\times\left\{\begin{array}{l}3,\ \text{width of}\\ \text{small square}\end{array}\right\}=\left\{\begin{array}{l}249\ \text{used up}\\ \text{in strips 240,}\\ \text{and in small}\\ \text{square 9,}\end{array}\right\}$$

∴ the square foot of 1849 = 43.

This may be abbreviated thus:

$$\begin{array}{rl} & 1849(43 \\ & 16 \\ 83) & \overline{249} \\ & \underline{249} \end{array}$$

∴ the square root of 1849 = 43.

ALTERNATIVE EXPLANATION

The operation employed above may also be suggested by analysing the product got by multiplying 43 by 43. Thus:

$$\begin{array}{rcr} 43 & = \ldots\ldots\ldots & 40+3 \\ \underline{43} & = \ldots\ldots\ldots & \underline{40+3} \\ 129 & = \ldots\ldots\ldots & 40\times 3+3^2 \\ \underline{1720} & = \ldots\ldots\ldots & \underline{40^2+40\times 3} \\ 1849 & = \ldots\ldots\ldots & 40^2+2\times 40\times 3+3^2 \\ & = \ldots\ldots\ldots & 40^2+(2\times 40+3)3. \end{array}$$

Using this form of the product 43×43, or 1849, we see:

1. That 40^2, or 1600, is the part corresponding to 40 or 4 tens of our square root;

2. That when this is taken we have (2×40×3) ×3, or 249 left;

3. That 2×40 gives us part of our divisor, and that the quotient of 240 by 2×40 suggests the other figure of the answer and the other term of the divisor.

Hence we have the different steps given in the *Ontario Public School Arithmetic*, pages 202 and 203.

Note: Care must be taken to begin at the decimal point when marking a number into groups, or periods, before finding the square root. For example, to find the square root of ·04 we mark off thus $\cdot\overline{04}$, and ·2 is seen to be the square root. To find the square root of ·4 we mark off thus, $\cdot\overline{4}$, and add a cipher thus, $\cdot\overline{40}$, and the first figure of the result is seen to be ·6, since the 6 is the highest integer which when squared is less than 40.

For Exercises, see *Ontario Public School Arithmetic.*

CHAPTER XX

MENSURATION

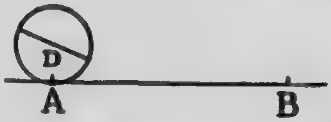
(A) PERIMETERS

GIVE to each member of the class a square and a rectangle cut out of cardboard. Draw attention to, (1) the relative lengths of adjacent sides in each, and (2) the character of the angles. Give the names *square* and *rectangle,* and have the pupils define each. Let the pupils draw other squares and rectangles on the black-board and in their exercise books, and give dimensions after measuring. Take strings or rulers and find the perimeter of the figures drawn.

Also find the perimeter of book, desk, table, schoolroom, etc., using strings when the sides are not an integral number of inches or feet.

Give the pupils circular cardboard discs or other circular objects such as wheels, cylindrical cans, covers, etc., of different diameters. Define the terms *circumference, diameter, radius.*

CIRCUMFERENCE OF CIRCLE



$$\frac{AB}{D} = \frac{\text{CIRCUMFERENCE}}{\text{DIAMETER}} = \frac{22}{7}$$

Place a mark on the circumference of a disc and roll the disc on a table, or on the floor, or the black-board. Mark the points on the table touched by the mark on the circumference and measure the distance between these points. This gives the length of the circumference AB. Now measure carefully the diameter of the disc used, D.

Divide the length of the circumference by the length of the diameter and carry the quotient to two places of decimals.

The circumference = 3·14 times the diameter,
= $\frac{22}{7}$ times the diameter (approximately).

This is often written thus:
Circumference $= \frac{22d}{7}$ where d is diameter; or
Circumference $= \frac{22}{7} \times 2r = \frac{44r}{7}$ where r is radius.

PROBLEMS

1. A metal disc is 7 in. in diameter; what is its circumference?

2. A cart wheel is 5 ft. in diameter; how long is its tire?

3. The front wheel of a wagon is 3 ft. 6 in. in diameter, and the hind wheel is 4 ft. in diameter. How often will each turn in going a mile? Which axle will wear faster? Why?

4. A circular race-course is 80 rods in diameter. What is the distance round it?

5. The tire of a wheel is 11 ft. long. What is the diameter of the wheel? About how long is a spoke?

6. You want a circular race-course just one mile long, what should be the radius of the circle?

7. Two circles have the same centre; the inner one has a diameter of 35 ft. and the outer a diameter of 70

ft. How much farther is it round the outer circle than round the inner?

8. A large window 5 ft. 10 in. wide and 8 ft. high has a semicircular top above the rectangular part. Make a drawing of the whole window and find its perimeter.

9. A belt runs between two wheels whose diameters are 35 in. and 7 in. respectively. If the large one turns 200 times a minute, how many turns does the small wheel make a minute?

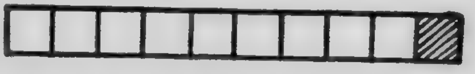
For other questions, see *Ontario Public School Arithmetic.*

(B) AREAS

RECTANGLES

Give to each member of the class a square inch. Those used in the primary class answer the purpose well. Have the pupils pass their fingers over the surface and note the area.

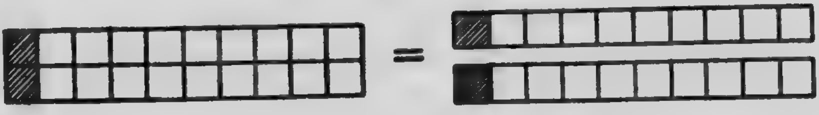
Give strips of paper or cardboard to each. Let the strips be one inch wide and 10 inches, say, in length. Have the pupils mark off the surface into square inches, thus:



Make a drawing on the black-board and reckon its area thus:

Area = 1 sq. in. × 10.

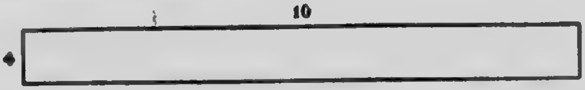
Next, give to each pupil a strip 10 inches by 2 inches and have it marked off as before into square inches. Next, cut the strip into two parts along the length. Make drawings on the board thus:



Area = 1 sq. in. × 10 × 2 = 20 sq. in.

Give the meaning of the word *dimensions,* and ask for the area of rectangles 4 inches wide by 10 inches long, etc., etc.

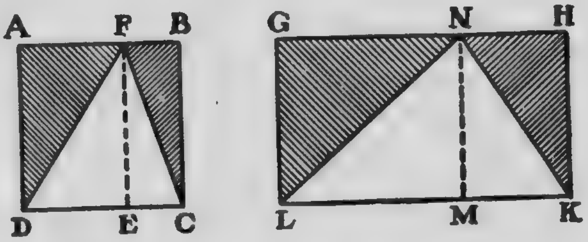
Encourage the making of diagrams with dimensions marked thereon, thus:



Area = 1 sq. in. × 4 × 10 = 40 sq. in.

For Exercises, see *Ontario Public School Arithmetic.*

TRIANGLES



Make a square and a rectangle out of cardboard. Note the altitudes AD, FE, BC, or GL, NM, HK, and the bases DC, and LK. The class will know the area of the square.

Measure of area of square = measure of base × measure of altitude,

or briefly, area = $b \times a$.

Similarly, measure of area of rectangle GLKH = measure of LK × measure of NM, or area = base × altitude.

Note, also, triangle FDC and square ADCB have the same base and altitude; also triangle NLK and rectangle GLKH have the same base and altitude;

and also that triangle FEC = triangle EBC
" FED = " FDA
" NMK = " NHK
" NML = " NGL

Now score GLKH along the lines NK and NL so that the triangles NGL and NHK may be turned back, leaving the triangle NLK, which is thus seen to be ½ the area of the rectangle.

That is, triangle NLK = ½ rectangle GLKH
= ½ base × altitude
= ½ $b \times a$

State the result generally thus:

The measure of the area of a triangle = ½ the measure of its base × the measure of its altitude;

or briefly, *area of triangle* = ½ $b \times a$.

PROBLEMS

1. The base of a triangle is 4 ft. and its altitude is 1½ ft. Find its area.

2. The base of a triangle is 4 ft. and its altitude 6 ft. Find its area.

3. Find the sum of the areas of two triangles whose bases are 4 ft. and 6 ft. respectively, and whose altitudes are each 2 ft.

Show the solutions worked separately. Show the solution of both as forming one question.

Thus: Measure of area of the first = ½ measure of base × measure of height,

$$= \tfrac{1}{2} \text{ of } 4 \times 2 = 4,$$

$$\therefore \text{area} = 4 \text{ sq. ft.}$$

$$\text{Similarly area of second} = 6 \text{ sq. ft.},$$

$$\therefore \text{area of both} = 10 \text{ sq. ft.}$$

Or, working the two parts together, since the altitude is the same we have:

Measure of area of both = ½ measure of sum of bases × measure of altitude,

$$= \tfrac{1}{2}\,(4+6)\times 2 = 10,$$

$$\therefore \text{area of both} = 10 \text{ sq. ft.}$$

4. Find the sum of the areas of these four triangles:
A triangle whose base is 6 ft. and altitude 4 ft.;
A triangle whose base is 8 ft. and altitude 4 ft.;
A triangle whose base is 9 ft. and altitude 4 ft.;
A triangle whose base is 7 ft. and altitude 4 ft.

Measure of sum of area = ½ measure of sum of bases × measure of altitude,

$$= \tfrac{1}{2}\,(6+8+9+7)\times 4,$$

$$= \tfrac{1}{2} \text{ of } 30\times 4 = 60,$$

$$\therefore \text{sum of areas} = 60 \text{ sq. ft.}$$

THE RIGHT-ANGLED TRIANGLE

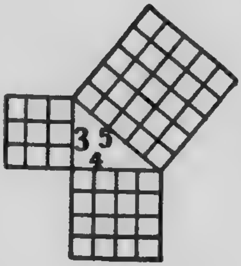
Have the pupils draw right-angled triangles, and point out in each the hypotenuse and the sides containing the right angle.

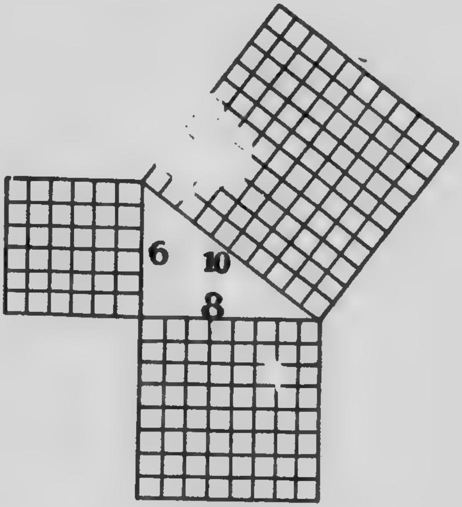
1. Have them draw a right-angled triangle with sides about the right angle 3 units and 4 units in length, and find by actual measurement the length of the hypotenuse.

2. Have them draw a right-angled triangle with sides about the right angle 6 units and 8 units in length, and measure the hypotenuse.

3. Have them find the hypotenuse of a right-angled triangle whose sides about the right angle are 5 units and 12 units in length.

Next, let them make these same triangles again and, on the sides of each triangle, construct squares outwardly and divide the squares thus drawn into small squares where the length of the side of a small square would be the length of a unit in the sides of the triangle.





In the first figure $3^2+4^2= 25$, and $5^2= 25$,

In the second figure $6^2+8^2=100$, and $10^2=100$.

1. From these and other similar illustrations it will be seen:

That *the square on the hypotenuse is equal to the sum of the squares on the other two sides;*

2. That, in consequence, if the two sides are given in length, the squares on these sides can be calculated and added, and thus the square on the hypotenuse may be found;

3. That if the area of the square on the hypotenuse is known, the length of one side may be found by extracting the square root of the measure of this area;

4. And, therefore, that when the two sides of a right-angled triangle are known, the third side, or hypotenuse, may be calculated.

It was found that, if the two sides of a right-angled triangle were 5 units and 12 units in length respectively, the third side was 13 units in length. But the length 13 units was found by measurement. It is now seen how it may be obtained by calculation, thus:

Length of one side $= 12$ units, therefore square on this side contains 144 units.

Length of other side $= 5$ units, therefore square on this side contains 25 units.

$\therefore$ the square on the hypotenuse contains $(144+25)$ or 169 units of area,

$\therefore$ the length of one side $= \sqrt{169} = 13$ units.

If the sides are denoted by b and p and the hypotenuse by h we may express our results thus:

$$1.\ b^2 + p^2 = h^2$$

$$\text{or } 2.\ h = \sqrt{b^2 + p^2}$$

$$\text{also we have } 3.\ b^2 = h^2 - p^2$$

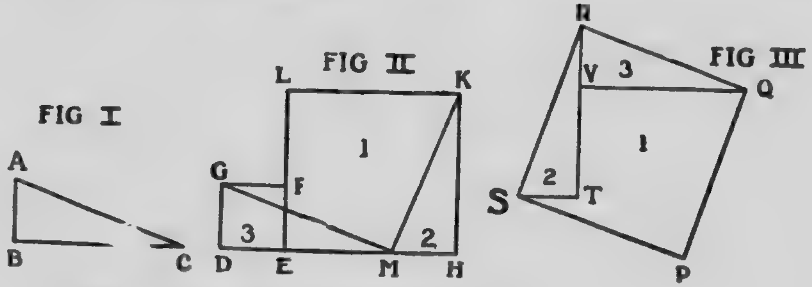
$$\therefore\ b = \sqrt{h^2 - p^2}$$

$$4.\ p^2 = h^2 - b^2$$

$$\therefore\ p = \sqrt{h^2 - p^2}.$$

Another General Proof

The fact that the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle of a right-angled triangle may be shown experimentally for any right-angled triangle by a method suggested by the following diagram:



ABC is any right-angled triangle.

GDEF and LEHK are equal to the squares on AB and BC of the triangle ABC.

MH is taken = AB, and therefore DM = BC.

Join MG and MK. Now if Figure II be cut out of stiff paper or cardboard and cut into three pieces marked 1, 2, 3, these pieces may be put together as in Figure III to form the large square, which is the square on the hypotenuse GM or AC.

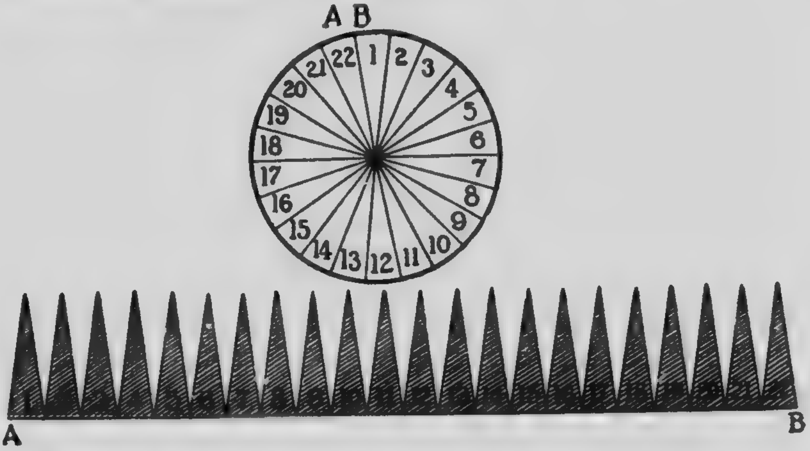
Therefore the square RSPQ is equal to the sum of the squares GDEF and LEHK.

That is, the square on the hypotenuse is equal to the sum of the squares on the other two sides. Or if cardboard be used, the square on the hypotenuse is found to weigh as much as the sum of the weights of the squares on the other two sides. And since the thickness is uniform, the area of the large square on the hypotenuse must be equal to the sum of the areas of the other two squares.

AREA OF A CIRCLE

Cut out of wood a circular disc ½ inch thick. Cut it into two equal parts, and afterwards cut from centre to circumference along many radii with a fine saw, thus dividing the disc into many small sectors. Place all the sectors together again as before, and tack round the circumference a flexible leather strap.

To find the area of the surface of the disc open out as shown in the figure, when it will appear that the area of the circle is equal to the sum of the areas of all the triangles standing on the strap.



Lead the pupils to observe:

1. That the area of the circle is equal to the combined areas of all the triangles;
2. That the altitudes of the triangles are the same, namely the radius of the circle;
3. That the bases of all the triangles when added make up the circumference of the circle, that is, $\frac{44}{7}r$.

Set down thus:

measure of area of circle = measure of area of all the triangles,

= $\frac{1}{2}$ of sum of measure of bases × measure of altitude,

= $\frac{1}{2}$ measure of circumference of circle × measure of radius of circle.

$= \frac{1}{2} \times \frac{44}{7} \times r \times r,$

$= \frac{22}{7} \times r^2$, that is, $\frac{22}{7}$ (measure of radius)^2.

For Exercises on the areas of circles, see *Ontario Public School Arithmetic.*

THE SURFACE OF A CYLINDER

Take any cylindrical solid. One of the cylindrical blocks used for notation will do. Observe its ends. What are they? Observe its curved surface.

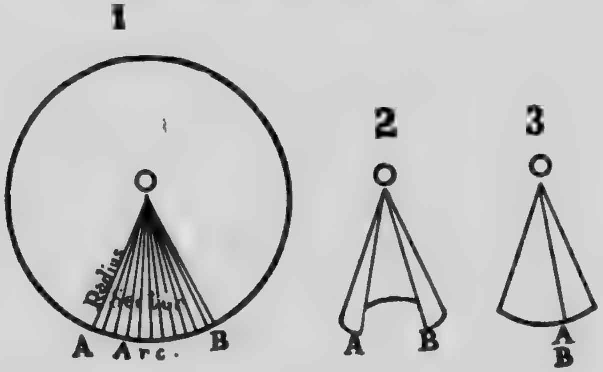
Cover the curved surface neatly with paper and, while the paper remains on the cylinder, write on that edge corresponding with the circumference of the end of the cylinder the words—"circumference of end of cylinder", and along the edge of the paper corresponding to the height, the words "height of cylinder". Now unroll the paper. Note that it forms a rectangle.

The measure of the area of the curved surface of the cylinder is the measure of the circumference of the end multiplied by the measure of the height.

Note also that the ends of the cylinder are circles, and that therefore the measure of the area of each $= \frac{22}{7}$ (measure of radius)^2.

For Exercises, see *Ontario Public School Arithmetic.*

THE SURFACE OF A CONE



To make a cone, cut out of a circular piece of stiff paper a sector, as shown in the accompanying diagrams. Bring the edges OA, OB together as in Figure 3, and, by means of a thin strip of paper, paste them in this position. Let the pupils note:

1. That the arc AB has become the circumference of the base of the cone;
2. That the radius of the sector has become the slant height of the cone;
3. That the area of the sector is the same as the area of the curved surface of the cone.

Just as, above, the circle was the sum of all the small triangles, so here the sector is the sum of the areas of a number of triangles into which it may be divided, the sum of whose bases is AB and whose height is OA, or the radius of the sector.

Therefore the measure of the area of the sector
= ½ the measure of AB multiplied by the measure OA,
= ½ the measure of the circumference of the base of the cone × the measure of the slant height of the cone,
∴ *the measure of the area of the curved surface of the cone*
= ½ *the measure of the circumference of its base multiplied by the measure of the slant height.*

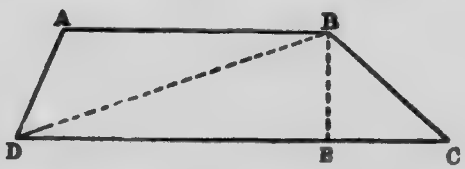
Note also, if the cone is solid, its base will be circular, and therefore its area is easily found.

PROBLEMS

1. Find the surface of a conical tent the diameter of whose base is 14 ft. and whose slant height is 15 ft.

2. A silo is 14 ft. in diameter. It is cylindrical to a height of 24 ft. Its roof is conical and its slant height is 16 ft. Find the entire outer surface of the silo.

AREA OF A TRAPEZIUM



Cut out of paper a quadrilateral having two sides parallel. Mark on it a line such as BE to show the perpendicular distance between its parallel sides. Cut it into two triangles as shown in the Figure, and place AB, one parallel side, in line with DC, the other parallel side.

Note.—1. That the altitude of each triangle is BE,

2. That the trapezium = the sum of the triangles,

$\therefore$ the measure of the trapezium

= the measure of the sum of the areas of the triangles,

= $\frac{1}{2}$ of the measure of the sum of the bases × the measure of the altitude,

$= \frac{1}{2}\ (DC + AB) \times BE$

$\frac{1}{2}$ the measure of the sum of the parallel sides of the trapezium × the measure of the perpendicular distance between them.

VOLUMES

RECTANGULAR SOLIDS

Give the members of the class a number of cubes, each a cubic inch.

Have them note the length of the edges, and the number and character of the faces.

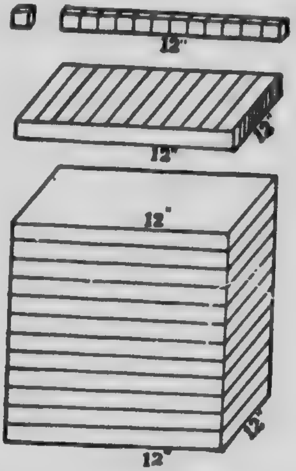
The amount of space occupied by this cube is 1 cu. in. and its volume is said to be 1 cu. in.

Next present a block 12 in. long, *1 in. wide,* 1 in. thick. How many cu. in. are found in it? 12.

Then what is its volume? 12 cu. in.

If the block were 12 in. long, *2 in. wide,* 1 in. thick, what would be its volume?

Next take the slab off the cubic foot found in schools, and let the pupils compare its volume with the volume of the rod taken first. Now present the cubic foot and let them compare its volume with that of the slab. Then give the problem: How many times must the cu. in. be taken to make up the cubic foot? What is the volume in cubic inches of a block 12 in. long, 12 in. wide, and 12 in. thick?



The result may be set down thus:

Unit of measurement = 1 cu. in.
Vol. of rod = 1 cu. in. × 12.
Vol. of slab = vol. of rod × 12
= 1 cu. in. × 12 × 12.
Vol. of large cube = vol. of slab × 12
= 1 cu. in. × 12 × 12 × 12
= 1728 cu. in.

Or briefly thus:

Volume of cube = 1 cu. in. × 12 × 12 × 12
= 1728 cu. in.

A cubic yard may be represented on the black-board, and its volume found similarly and expressed thus:

Volume = 1 cu. ft. × 3 × 3 × 3 = 27 cu. ft.

After the dimensions of a cord are given as 8 ft. long, 4 ft. wide, 4 ft. high, its volume may be found thus:

Volume of cord = 1 cu. ft. × 4 × 4 × 8 = 128 cu. ft.

PROBLEMS

1. A block of wood is 36 in. long, 12 in. wide, and 8 in. deep. How many cu. in. are in it?

(Encourage the class to make diagrams placing thereon the dimensions. Volume = 1 cu. in. × 8 × 12 × 36 = 3456 cu. in.)

2. What is the volume of a rectangular solid (prism) with a base 6 inches square if its altitude is 4 in.?

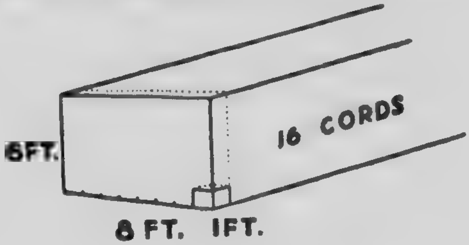
3. How many bushels will a bin 8 ft. square and 9 ft. high hold? (2218 cu. in. = 1 bu.)

4. Find the value at 65c. per bu. of the wheat that would fill a bin 15 ft. square and 12 ft. deep.

Illustration

To find the length of a rectangular solid, its volume, height, and width being given.

Example: A pile of wood contains 16 cords. Its end dimensions are 6 ft. high, and 8 ft. wide; find its length.



Represent the pile as in the diagram.

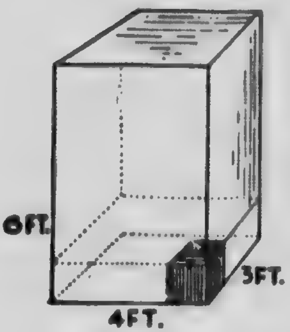
Find the number of cu. ft. in 16 cords. This is 128 cu. ft. × 16 = 2048 cu. ft.

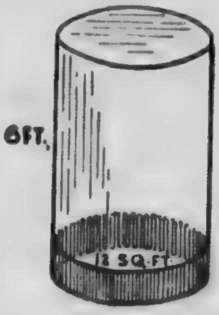
When 1 ft. of length of pile is used, 1 cu. ft. × 6 × 8, or 48 cu. ft. is accounted for.

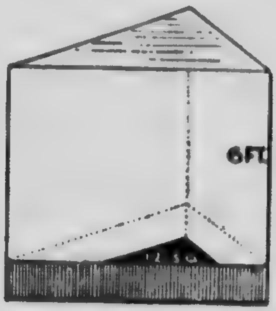
Since 48 cu. ft. gives 1 ft. in length, 2048 cu. ft. will give $\frac{2048}{48}$ times 1 ft. in length.

∴ the pile is 42⅔ ft. in length.

VOLUME OF CYLINDER AND PRISMS







Represent on the black-board as in the accompanying diagram:

1. A rectangular prism with base 4 ft. by 3 ft. and height 6 ft.;

2. A cylinder with the same base area, namely, 12 sq. ft., and the same height, namely, 6 ft.;

3. A triangular prism with base area 12 sq. ft. and height 6 ft.

If a slice 1 ft. thick be cut off the base of each, the part cut off will be such that:

1. In the case of the rectangular prism the volume will be 1 cu. ft. × 4 × 3 = 12 cu. ft.

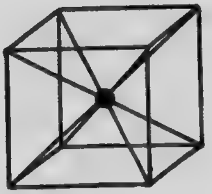
2. In the case of the others, since the area of their bases is 12 sq. ft., each sq. ft. may be regarded as the end of a cu. ft., and the number of cu. ft. in the slice will be the same as the number of sq. ft. in the end, and that is 12. Therefore, the volume of each slice is 12 cu. ft.

And since there are in each of the solids 6 such slices the total volume of each is 12 cu. ft. × 6 = 72 cu. ft.

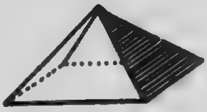
The measure of the volume of either a prism or a cylinder is thus seen to be *the measure of the area of the end multiplied by the measure of the height.*

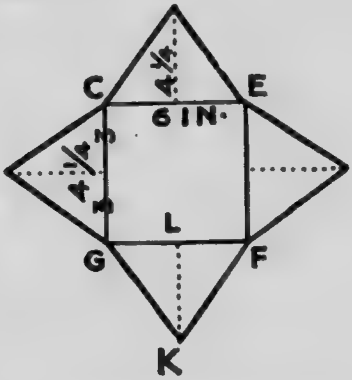
VOLUME OF PYRAMID

A



B





Take a piece of stiff cardboard. Mark off on it a centre 6 inches square and wings whose points are 4¼ inches from the sides of the square centre. Score the paper on the lines of the square and turn up the points to meet, as shown in Figure B.

Place six of these together and they form a 6-in. cube, as shown in Figure A.

Each pyramid will be 3 in. high, that is, ½ the height of the cube, and will have a base whose area is 36 sq. in.

Now the volume of the cube is 6 cu. in. × 6 × 6 = 216 cu. in., therefore the volume of one pyramid is $\frac{1}{6}$ of 216 cu. in. = 36 cu. in.

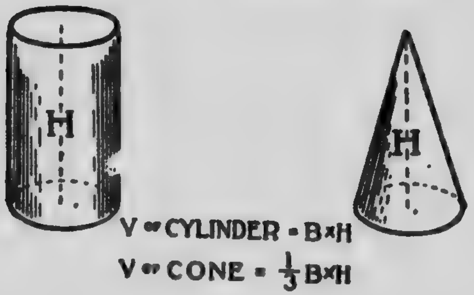
Thus the measure of the volume of the pyramid =

$$\frac{\text{the measure of the area of the base} \times \text{the measure of the height}}{3}$$

=36 cu. in.

Hence if volume = measure of volume of pyramid,
and b = measure of base of pyramid,
and h = measure of height of pyramid,
volume = $\frac{1}{3}$ of $b \times h$.

VOLUME OF RIGHT CONE



If a hollow cylinder and cone be obtained having equal bases and of the same height, it will be found that if the cone be filled with water and the water be emptied into the cylinder, this may be done three times before the cylinder is filled, that is, the volume of the cylinder is three times that of the cone.

But the measure of the volume of the cylinder equals the measure of the base multiplied by the measure of the height.

$\therefore$ Measure of volume of cone

$$=\frac{\text{measure of base} \times \text{measure of height}}{3},$$

or, briefly, $V = \frac{1}{3}$ of $b \times h$.

CHAPTER XXI

PROBLEMS FOR PRIMARY CLASSES

SECTION A

I. Use apparatus described on page 28.

(*a*) 1. Put 2 groups each of 2 splints in 2 holes.
2. Put 3 groups each of 3 splints in 3 holes.
3. Put 4 groups each of 4 splints in 4 holes.
4. Put 5 groups each of 5 splints in 5 holes.
5. Put 6 groups each of 6 splints in 6 holes.
6. Put 7 groups each of 7 splints in 7 holes.
7. Put 8 groups each of 8 splints in 8 holes.
8. Put 9 groups each of 9 splints in 9 holes.

(Note: Each arrangement is to be shown to the teacher.)

(*b*) 1. Put 1 splint into 1st hole, 2 splints into 2nd hole.
2. Put 3 splints into 3rd hole, 4 splints into 4th hole.
3. Put 5 splints into 5th hole, 6 splints into 6th hole.
4. Put 7 splints into 7th hole, 8 splints into 8th hole.
5. Put 9 splints into 9th hole.

(Note: This is to be kept to show the teacher.)

II. Make the groups in the picture on page 231 and set down the proper figure opposite each:

III. Make dots opposite the following figures:
1, 3, 4, 2, 7, 6, 9, 8, 5.

IV. Make first one object, then a group made by adding two objects, then another by adding two more, etc., and set down the proper figure for each group.

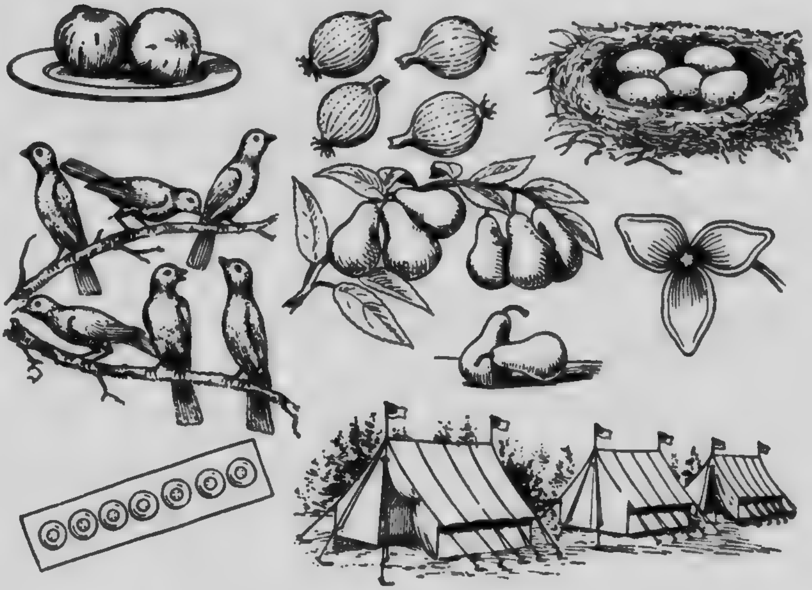
V. Make a group of 2 objects, then another greater by two objects, etc., and set down the proper figure opposite each group.

VI. Count to nine and set down the figures as you count.

VII. Arrange the following figures: 2, 6, 1, 5, 3, 4, 7, 9, 8,

(1) So that the smallest comes first, the next to the smallest, next, and so on.

(2) So that the largest comes first, the next to the largest, next, and so on.



VIII. Set down 6 three times in a vertical line, and to the right of each figure set down 6 objects. Then divide each group into two groups by a straight line, and set down in figures the number found in each part, thus:

. .
for 5 . 2 and 3
. .

. .
5 . 1 and 4
. .

IX. Deal with 7, 8, 9, as with 6.

X. Set down all the pairs of numbers which make, 5, 6, 7, 8, 9.

XI. Make 2 lines, each 2 inches long.
Make 3 lines, each 3 inches long.
Make 4 lines, each 4 inches long.
Make 5 lines, each 5 inches long.
Make 6 lines, each 6 inches long.

XII. Make a pyramid of lines so that the top one is 1 inch, the next 2 inches, ... 6 inches.

XIII. Take only 7 splints and set them into the holes in the number board in any way, at the same time setting down in figures the numbers in the holes, for example:

2, 2, 2, 1.

Now take out and set them into holes in another way, and set down in figures the numbers in the holes as before, etc.

Keep all your figures.

XIV. Before going on to the questions in addition where the sum is more than 10, many questions such as the following should be given to secure accuracy and rapidity:

Examples where the sum is 7:

1	2	3	4	5	6
6	5	4	3	2	1
—	—	—	—	—	—
1	2	3	4	5	5
1	1	1	1	1	1
5	4	3	2	1	1
—	—	—	—	—	—

XV. Write the numbers from:

(1) 1 to 20; 1 to 30; 1 to 50; 1 to 100.
(2) 13 to 27; 15 to 36; 26 to 38; 64 to 89.

XVI. Write the numbers by fives from:

(1) 5 to 50; from 5 to 100.
(2) 15 to 35; 35 to 85; 75 to 100.

XVII. Arrange the following numbers so that the largest comes first and so on: 13, 41, 78, 54, 91, 84, 29, 63.

XVIII. Write the numbers by tens from:

(1) 10 to 50; 10 to 60; 10 to 100.
(2) 1 to 31; 1 to 51; 1 to 91.
(3) 2 to 22; 2 to 62; 2 to 82.
(4) 3 to 33; 3 to 53; 3 to 93.

. . . .

(9) 8 to 88; 9 to 99.

XIX. Write by twos from 2 to 100.
Write by twos from 1 to 99.

XX. Write down all the numbers between 0 and 100:

(1) Ending in 1.
(2) Ending in 2.

. . . . etc.

XXI. What is the sum of (1) 10 and 1? 10 and 2? 10 and 3? 10 and 6? 10 and 8? 10 and 9?
(2) What with 10 makes 14? 16? 18? 19?
(3) What with 20 makes 24? 27? 28? 29?
(4) What with 60 makes 67? 69? 66?
(5) What with 7 makes 17? 27? 37? 87?
(6) What with 8 makes 18? 38? 68? 88?

XXII. (1) How many tens in 30? 40? 60? 80? 90?
(2) How many tens in 27? 34? 63? 81? 95?

XXIII. How much greater is 18 than 8? 16 than 6? 20 than 18? 26 than 16? 38 than 28? 66 than 56?

SECTION B

I. (*a*) Add the following:

5	15	35	65	85	75	
5	5	5	5	5	5	(oral)
—	—	—	—	—	—	

(*b*) Add the columns, beginning at the foot of each:

5	5	4	7	6	3	6	7
5	5	5	5	5	5	5	5
—	5	5	5	5	5	5	5
	5	5	5	5	5	5	5
	—	5	5	5	5	5	5
		5	5	40	60	70	50
		5	5	—	—	—	—
		—	5				
			5				
			—				

II. (*a*) Add:

9	19	29	49	69	79	59	
1	1	1	1	1	1	1	(oral)
—	—	—	—	—	—	—	

(*b*) Add at seats:

9	9	2	7	3	6	4	7	8
1	1	9	1	9	1	5	1	1
9	9	1	9	1	9	5	9	9
1	1	5	1	5	9	9	9	9
—	9	5	5	5	1	1	1	1
	1	—	5	30	40	20	60	5
	—		—	—	—	—	—	5
								30
								—

III. (*a*) Add:

8	18	38	68	48	78	28	
2	2	2	2	2	2	2	(oral)
—	—	—	—	—	—	—	

(*b*) Add at seats:

6	4	6	4	9	8	8	8	6
2	5	9	8	1	2	2	2	8
8	5	1	2	8	2	5	9	2
8	8	8	8	2	8	5	1	5
2	2	2	2	20	30	40	60	5
—	—	—	—	—	—	—	—	8
26	2	26	9	40				2
								70
								—
								106

IV. (*a*) Add:

7	17	37	57	87	67	
3	3	3	3	3	3	(oral)
—	—	—	—	—	—	

(*b*) Add at seats:

7	4	7	9	9	8	8	6
3	3	3	7	1	2	2	7
7	7	5	2	7	7	7	3
3	7	5	7	3	3	3	8
—	3	7	3	20	30	50	2
	—	3	—	—	—	—	7
		—					3
							60
							—

V. (*a*) Add:

6	16	36	66	76	86	56	
4	4	4	4	4	4	4	(oral)
—	—	—	—	—	—	—	

(*b*) Add at seats:

5	2	3	9	6	5	1
5	9	6	1	4	5	9
6	1	4	8	8	8	6
4	6	8	2	2	2	4
6	4	2	6	6	6	80
4	6	6	4	4	4	—
—	4	4	30	60	70	
3	—	30	—	—	—	
	4	—				

VI. Add at seats:

5	3	4	2	2	3	5	2
2	7	1	2	3	2	8	3
4	2	5	2	5	3	2	3
4	3	4	2	2	2	2	4
5	5	3	7	4	5	4	3
2	4	3	3	4	5	2	8
3	3	4	1	1	2	2	2
5	3	7	5	3	4	7	9
1	1	3	4	3	4	3	1
4	2	3	1	3	1	7	1
5	3	5	3	8	3	3	2
5	4	2	6	2	2	7	3
					4	3	4

VII. Add at seats:

4	2	4	4	4	6	6	4
4	2	2	4	2	4	2	1
2	4	2	2	3	1	3	4
3	1	9	4	5	4	5	3
5	4	1	1	4	4	2	2
1	5	1	3	2	1	3	9
4	3	2	3	4	1	2	1
3	5	3	3	1	4	2	7
2	2	4	2	6	3	3	3
6	1	1	2	3	2	2	1
4	2	4	2	1	2	4	2
2	3	5	2	2	3	4	7
3	4		2	7	5		
5							

VIII. Add at seats:

2	3	1	32	10	44	27
5	3	4	24	14	42	42
1	2	4	41	85	14	45
3	5	1	44	31	54	95
3	3	3	25	29	41	11
3	3	3	21	52	77	84
1	3	3	22	12	32	24
3	1	1	23	32	11	71
3	4	2	24	32	76	33
3	2	3	21	32	23	84
2	3	3	23	32	21	23
8	1	1	36	48	23	27
					26	43
31						

SECTION C

The following problems furnish exercises for the combinations, or addition facts, in the order of the endings, 9, 8, 7, 6, 5, 4, 3, 2, and 1.

The problems given below furnish exercises on the combinations, or addition facts, of the numbers from 10 to 20 taken in their natural sequence, 11, 12, 13, etc.

The teacher will not, however, find it difficult to make the problems in either Section supplement those in the other.

It should not be necessary to point out that the exercises given are intended as suggestions only, and are not to be understood as giving a full treatment of combinations.

I. (*a*) Add:

9	9	9	9	9	9	9	9	
9	29	19	39	49	69	89	79	(oral)

(*b*) Add at seats:

9	9	7	9	8	1	9	9	72
9	9	9	1	9	9	9	9	29
2	9	1	9	9	9	6	2	99
9	1	9	9	2	3	4	9	96
9	9	9	2	9	9	2	9	74
	9	6	9	9	9	9	8	86
	2	4	9	3	5	9	2	64
	8			7	5			

II. (*a*) Add:

8	8	8	8	8	8	8	8	
8	18	38	28	58	68	88	78	(oral)

(*b*) Add at seats:

4	9	8	3	8	8	8	74	52	88
8	8	9	9	9	9	8	28	29	88
8	8	9	1	9	9	4	98	99	24
7	4	4	4	8	4	8	99	98	98
3	8	8	8	2	8	9	61	52	98
8	8	9	8	4	8	9	59	57	94
2	7	9	4	6	4	2	61	68	18
7	3	56	8	46	8	9			58
8	55		8		8	9			
50			53		76	66			

III. (*a*) Add:

7	7	7	7	7	7	87	27	
7	27	17	37	57	87	7	7	(oral)
—	—	—	—	—	—	—	—	

(*b*) Add at seats:

5	7	6	1	6	7	6	97	87
5	6	7	8	4	7	7	60	87
6	7	7	2	8	6	7	76	64
7	7	4	4	8	7	2	77	78
7	8	8	8	6	7	9	87	78
6	2	8	9	7	2	9	28	92
7	8	2	9	7	9	4	72	19
7	2	9	6	2	29	8	—	59
—	—	9	7	9	—	8		—
		—	7	9		18		
			—	—		—		

IV. (*a*) Add:

6	6	6	6	6	6	6	
6	16	36	56	86	66	76	(oral)
—	—	—	—	—	—	—	

(*b*) Add at seats:

7	6	1	6	6	1	1	1	7	76	78
8	8	6	6	6	6	8	8	8	86	86
6	8	6	2	2	6	6	6	6	68	66
6	6	8	9	9	2	6	6	6	66	66
8	7	6	9	9	9	8	2	8	76	74
6	7	6	2	2	9	6	9	6	36	36
6	6	8	9	9	2	6	9	6	64	64
8	4	6	9	9	9	8	6	8	—	—
6	—	6	—	—	9	6	7	6		
6		—			—	8	7	6		
—						8	—	—		
						—				

7

V. (*a*) Add:

4	4	4	4	4	4	
14	24	34	64	74	84	(oral)
—	—	—	—	—	—	

(*b*) Add at seats:

7	9	8	7	2	6	8	84	89	87
4	9	4	2	4	8	4	47	49	92
8	2	4	4	4	8	4	77	42	94
8	4	8	7	8	2	8	72	84	24
2	4	6	7	6	4	6	89	64	44
4	2	6	6	8	4	8	69	82	48
4	4	8	7	8	2	8	66	84	78
2	7	6	7	2	4	4	64	74	—
9	7	6	—	9	7	8	—	—	
9	—	—	[illegible]	9	7	8			
—	48	56		—	—	—			
57				60	52				

VI. (*a*) Add:

3	3	3	3	3	3	3	
3	13	23	63	83	93	53	(oral)

(*b*) Add:

2	2	2	2	2	2	2	
2	12	32	62	72	82	92	(oral)

(*c*) Add at seats:

6	6	6	6	6	4	7	2	43	64
3	8	8	3	3	2	2	2	83	32
3	8	3	3	8	2	4	4	42	36
4	4	4	8	2	2	2	2	24	26
3	3	3	6	4	4	2	2	12	48
3	3	3	6	4	4	8	4	11	46
4	2	4	8	2	2	6	8	91	76
3	9	8	6	4	4	3	8		
3	9	8	6	7	4	3			
				7					

52

VII. (*a*) Add:

8	8	8	8	8	8	8	
1	21	41	61	71	91	81	(oral)

(*b*) Add:

7	7	7	7	7	7	7	
72	22	32	62	42	92	52	(oral)

(*c*) Add at seats:

6	8	8	6	98	9	9	9	91	12
8	9	1	8	19	1	7	7	27	27
9	1	8	9	48	9	2	2	96	72
1	4	6	8	21	7	4	1	96	19
9	4	6	1	62	2	8	7	48	21
1	8	2	8	89	1	4	2	86	74
9	6	4	2	49	7	7	2	46	47
8	6	4	7		2	7	9		37
1			3				9		
52	46	59	52						

VIII. (*a*) Add:

5	5	5	4	4	4	4	
4	14	34	65	75	95	85	(oral)

(*b*) Add:

3	3	3	6	6	6	
6	16	26	43	53	73	(oral)
—	—	—	—	—	—	

(*c*) Add at seats:

9	2	4	1	94	4	9	1	37	28
5	9	4	5	45	8	3	3	82	99
4	4	9	4	51	9	3	6	91	47
2	5	5	1	14	6	3	1	73	56
9	1	7	4	45	3	2	5	28	16
5	4	7	5	55	2	9	2	19	78
4	5	8	5	85	9	3	6	36	26
—	5	6	5	—	6	3	6	33	56
	5	6	—		3	3	8	—	—
	—	—			—	—	6		
							6		
							—		

IX. (*a*) Add:

7	7	7	7	7	7	7	
11	21	61	81	21	31	61	(oral)
—	—	—	—	—	—	—	

(*b*) Add:

6	6	6	6	6	6	2	
2	22	32	82	72	92	66	(oral)
—	—	—	—	—	—		

(*c*) Add:

5	5	5	5	3	3	3	
3	23	48	63	75	45	85	(oral)
—	—	—	—	—	—	—	

(*d*) Add at seats:

1	8	8	9	8	6	88	69	67
7	6	6	5	9	8	58	87	81
2	8	2	4	6	7	32	52	32
2	7	2	2	3	1	22	31	55
7	1	6	6	2	2	47	25	23
1	4	2	2	4	5	21	64	42
2	8	2	2	4	3	—	—	42
7	8	6	7	2	2			86
1	—	2	1	5	6			—
—		—	—	3	2			
				—	—			

X. (*a*) Add:

6	6	22	5	5	5	
11	31, etc.	5	32	62	92, etc.	(oral)
—	—	—	—	—	—	

(*b*) **Add at seats:**

9	3	4	9	4	7	34	74	24	35
2	6	7	8	8	3	27	67	37	27
6	1	2	1	4	2	52	75	51	56
1	1	5	6	7	5	35	26	18	11
3	2	3	7	5	3	26	58	74	74
6	6	5	2	6	5	57	29	22	28
1	1	2	5	8	2	15	49	16	24
3	4	2	3	4	6	62	—	66	54
6	8	9	5	7	7	—		—	—
1	8	9	2	7	7				
—	—	—	—	—	—				
57	40	[illegible]	[illegible]						

(*c*) **Add:**

3	3	4	8	8	9	9	9	
14	64	73, etc.	29	39	69	58	48, etc.	(oral)
—	—	—	—	—	—	—	—	

(*d*) **Add at seats:**

6	3	7	7	4	7	3	7	7	7	1
6	3	8	9	7	3	3	5	8	1	8
8	1	9	2	4	4	3	2	9	8	9
8	2	8	8	3	3	9	3	3	5	3
9	4	3	2	6	8	4	9	5	3	9
2	3	6	5	7	7	7	4	6	2	8
9	3	1	8	8	2	5	7	8	4	3
9	8	3	2	5	1	2	8	9	7	9
—	9	5	7	4	6	6	9	9	7	9
	—	2	1	—	3	4	—	—	—	9
		—	—	43	—	—				—
		5								

XI. (*a*) **Add:**

5	1	2	2	4	7	7	
31	65, etc.	24	34	62, etc.	9	29, etc.	(oral)
—	—	—	—	—	—	—	

(*b*) **Add at seats:**

2	4	4	3	6	97	67	22	32	75
6	6	4	3	4	25	94	22	93	43
5	4	2	4	9	34	93	47	53	72
1	6	6	3	9	71	53	79	34	99
4	2	7	1	9	98	31	92	48	15
3	4	9	9	9	61	23	15	48	74
3	4	4	9	6	—	66	63	—	—
6	4	7	4	4	96	—	—		
4	2	9	7	—					
—	—	—	6	[illegible]					
			—						
			49						

XII. (*a*) Add:

7	7	7	8	8	6	6	6	9	
18	38	28, etc.	67	87	19	39	89	86	(oral)
—	—	—	—	—	—	—	—	—	

(*b*) Add at seats:

7	5	7	7	5	87	75	99	46	55
2	7	5	5	4	95	77	93	88	37
9	8	7	7	1	22	58	53	78	28
6	5	8	5	5	43	95	74	54	14
5	7	5	3	3	45	63	85	94	37
7	8	2	2	2	82	72	81	82	29
8	5	3	6	2	—	—	—	—	—
5	6	5	2	4					
7	9	7	2	7					
8	—	8	8	7					
—		—	—	—					

XIII. (*a*) Add:

5	5	5	9	9	6	6	6	6	
19	29	69	75	85, etc.	18	38	68	48	(oral)
—	—	—	—	—	—	—	—	—	

(*b*) Add at seats:

4	9	6	6	4	9	54	96	95	53
8	5	5	6	9	6	18	78	59	57
1	8	[illegible]	4	6	8	96	42	76	28
4	1	4	6	5	6	52	94	85	45
9	2	9	8	1	6	64	58	29	86
9	2	5	2	9	2	89	76	46	39
9	8	6	4	5	2	85	—	—	—
9	4	9	9	2	2	—			
9	7	5	5	4	4				
—	7	—	—	5	7				
	—			9	7				
				—	—				

XIV. (*a*) Add:

3	3	3	9	9	8	8	8	4	4
9	19	39	43	63, etc.	14	24	84	48	68
—	—	—	—	—	—	—	—	—	—

(*b*) Add:

7	7	7	5	5	5	
5	15	45	77	87	67, etc.	(oral)
—	—	—	—	—	—	

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(*c*) Add at seats:

2	2	9	7	9	9	7	64	33	92
3	3	3	9	8	9	4	99	89	98
7	7	2	9	5	8	8	83	98	34
9	3	[illegible]	2	7	8	8	18	38	98
3	7	8	5	8	2	8	26	32	47
8	9	2	7	9	5	2	36	37	75
9	3	4	8	9	7	5	—	25	—
3	8	8	7	2	7	7		—	
6	3	4	5	5	7	8			
4	9	—	—	7	6	7			
—	—			—	—	5			
						—			
						67			

XV. (*a*) Add:

9	9	9	2	2	7	7	7	4	4
12	32	62	49	79, etc.	14	44	64	87	57, etc.
—	—	—	—	—	—	—	—	—	—

(*b*) Add:

18	28	68	3	3	3	6	6	6	5	5
3	3	3	58	78	18	15	65	35	66	86 (oral)
—	—	—	—	—	—	—	—	—	—	

(*c*) Add at seats:

2	6	5	9	4	5	5	20	59	90
9	2	4	5	5	2	5	99	48	99
9	8	7	4	8	8	6	12	93	88
9	9	9	4	8	2	9	39	79	72
2	2	2	7	9	8	8	86	48	48
2	6	8	2	3	2	2	75	63	42
9	4	2	7	9	9	3	36	—	39
9	6	8	7	8	8	9	64		—
—	7	7	7	3	2	9	—		
	7	4	7	8	8	—			
	—	—	—	—	7	54			
	57				—				

XVI. Typical problems in addition:

1. Harry paid 4 cents for a mask, 8 cents for a wig, and 6 cents for a horn. How much did he pay for all?

2. Mary earned 25 cents on Monday, 36 cents on Tuesday, and on Wednesday as much as on Monday and Tuesday. How much did she earn on Wednesday? How much on the three days?

3. At a party, there were 17 boys and 22 girls. How many children were there at the party?

103
94

13
9

4. If 44 boys rode to a picnic in one car, and 52 in another, how many rode in both cars?

5. John has 22 rare stamps, and James has 26 more than John. How many have both?

6. George spent 22 cents for a bat, 50 cents for a ball, and 75 cents for a glove. How much did he spend in all?

7. Willie had 15 cents left after spending 50 cents for a fishing-rod and 13 cents for hooks. How much had he at first?

8. Tom spent 24 cents for nuts, 22 cents for grapes, and 37 cents for figs. How much did he spend for all?

9. Edith spent, for her party, 15 cents for lemons, 12 cents for sugar, 20 cents for cake, and 45 cents for ice-cream. What did the party cost?

10. John had 37 marbles; his uncle gave him 36 more and, on his way to school he bought 48. How many had he in all? 121

SECTION D

I. (*a*) Add:

5	5	5	5	5	5	
6	16	36	86	46	26	(oral)

(*b*) Add:

6	6	6	6	6	6	
5	15	35	95	45	55	(oral)

c) Add at seats:

31
24
31

3	2	4	4	2	6	4	4	3
6	3	5	2	3	5	6	5	3
4	2	6	7	4	3	4	6	3
6	4	5	5	6	6	5	3	6
2	5	4	2	5	6	2	2	4
2	2	5	3	2	1	3	4	6
6	3	2	6	4	3	5	6	5
4	6	3	4	3	6	1	4	6
5	4	6	5	6	4	4	6	3
6	6	4	6	4	5	6	3	6
8	1	6	7	5	2	2	2	5
2	4	5	3	6	4	3	30	40
—	—	—	—	—	—	—	—	—

II. (*a*) Add:

7	7	7	7	7	7	
4	14	44	64	84	34	(oral)

(*b*) Add:

4	4	4	4	4	4	
7	17	57	27	67	97	(oral)

(*c*) **Add at seats:**

3	3	2	3	4	5	64	43	34
5	5	5	5	2	5	11	63	36
4	4	4	4	4	7	45	66	51
4	6	3	6	6	4	44	54	54
2	4	4	4	4	4	57	35	75
4	3	9	1	4	5	21	35	35
6	3	4	5	1	4	33	37	45
7	7	6	7	5	2	75	44	37
3	3	7	4	6	4	36	66	33
4	4	3	3	4	5	74	73	74
7	3	7	7	4	6	34	35	32
	4	4		7		67	66	65

III. (*a*) **Add:**

3	3	3	3	3	3	
8	18	48	98	58	38	(oral)

(*b*) **Add:**

8	8	8	8	8	8	
3	53	23	63	43	13	(oral)

(*c*) **Add at seats:**

4	3	4	4	83	34
4	7	7	8	34	73
3	3	3	2	16	13
6	6	3	7	34	55
3	4	7	3	38	41
3	3	3	3	22	33
4	7	3	3	86	73
3	8	4	4	24	52
7	2	8	6	73	47
3	3	2	4	32	83
4	3	8	3	33	23
4	5	3	8	43	78

23 5

IV. (*a*) **Add:**

9	9	9	9	9	9	
2	12	52	72	42	32	(oral)

(*b*) **Add:**

2	2	2	2	2	2	
9	49	79	19	59	89	(oral)

(*c*) Add at seats:

2	4	2	3	44	45	33	42
3	4	3	7	34	24	55	64
4	2	5	3	77	46	24	42
2	3	8	9	63	84	33	63
4	3	2	1	42	23	53	54
4	2	7	8	83	24	76	22
2	9	3	2	25	33	34	34
4	1	6	6	26	57	69	66
4	2	4	4	54	24	41	44
9	4	2	4	38	39	32	33
2	5	9	7	73	53	49	48
					58		

V. (*a*) Add:

6	6	6	6	6	6	
6	16	86	56	36	96	(oral)

(*b*) Add:

7	7	7	7	7	7	
5	25	95	45	65	35	(oral)

(*c*) Add:

5	5	5	5	5	5	
7	97	47	17	67	37	(oral)

(*d*) Add at seats:

4	3	4	3	3	2	4
3	4	9	4	4	3	3
6	6	6	6	7	4	6
4	6	1	4	3	7	7
6	3	3	7	6	3	3
1	5	6	3	4	6	5
3	6	4	5	5	4	5
6	3	6	1	5	5	5
4	1	5	4	5	6	1
6	6	3	5	4	7	3
6	3	3	7	3	4	3
59	3	5	4			
			6			
	49					

VI. (*a*) Add:

4	4	4	4	4	4	
8	28	68	48	98	38	(oral)

(*b*) Add:

8	8	8	8	8	8	
4	34	84	64	14	54	(oral)

(*c*) **Add at seats:**

8	3	2
4	3	4
5	4	2
8	6	4
8	7	6
4	8	4
2	4	3
4	1	3
4	5	8
8	4	2
4	5	8
6	3	4

VII. (*a*) **Add:**

3	3	3	3	3	3	
9	29	89	39	59	19	(oral)

(*b*) **Add:**

9	9	9	9	9	9	
3	43	83	23	13	73	(oral)

(*c*) **Add at seats:**

4	4	6	66	43	45	42
9	4	4	44	35	65	43
1	3	9	57	44	44	43
3	3	1	53	37	42	64
7	4	8	64	33	64	56
4	9	2	46	36	56	54
6	1	9	78	44	22	47
6	3	1	32	63	32	33
4	5	6	39	43	56	35
3	2	4	43	43	54	65
9	9	5	52	43	46	44
8	3	7			36	78
2						

VIII. (*a*) **Add:**

6	6	6	6	6	
7	47	27	87	37	(oral)

(*b*) **Add:**

7	7	7	7	7	7	
6	26	56	16	76	46	(oral)

(*c*) **Add at seats:**

3	3	3	2
4	4	2	4
7	7	4	7
5	5	6	5

(*c*) **Add at seats** (*Continued*):

4	2	6	6
4	3	4	2
6	8	5	2
7	6	6	2
4	4	3	9
3	7	7	8
7	3	2	8
6	8	4	4
—	—	—	—

IX. (*a*) **Add:**

5	8	4	9	
8, etc.;	5, etc.;	9, etc.;	4, etc.	(oral)

(*b*) **Add at seats:**

6	3	2	3	4	2	53	43	63	52
5	4	3	8	7	6	53	54	43	74
4	8	4	5	2	5	76	58	83	46
4	5	5	4	4	7	34	72	27	32
3	2	4	4	1	2	99	47	96	32
4	3	4	5	2	6	21	63	83	37
5	5	3	5	3	3	38	39	24	63
5	2	4	6	7	9	34	63	58	48
8	3	5	2	3	1	45	29	52	74
2	5	2	2	4	4	53	25	47	63
8	4	3	3	9	5	47	84	33	23
5	4	5	9	—	4	86	44	39	45
—	—	8	—		—	—	49	74	—
		—					—	—	

X. (*a*) **Add:**

7	7	6	6	8	8	
7	37, etc.;	8	48, etc.;	6	56, etc.	(oral)

(*b*) **Add at seats:**

2	8	4	3	4	4	4	2
6	4	7	2	3	3	4	3
3	4	4	2	2	2	6	5
5	8	5	6	3	3	4	6
4	7	7	4	6	6	4	4
7	6	6	4	1	1	3	4
8	7	3	6	3	3	3	3
7	2	4	3	7	7	8	3
6	3	4	4	3	8	2	6
7	5	7	3	6	6	6	4
7	7	5	4	4	4	4	6
7	4	4	8	4	4	6	3
—	3	8	6	—	—	8	5
	—	—	—			—	—

XI. (*a*) Add:

5	5	9	9
9	59, etc.;	5	65, etc. (oral)

(*b*) **Add at seats:**

2	2	9	37	22	29
3	5	5	33	43	31
1	5	5	46	55	68
4	7	5	51	45	46
5	3	2	33	66	73
5	5	3	35	44	33
6	1	9	72	97	57
6	4	1	33	16	16
4	5	7	86	67	43
4	3	8	24	42	84
5	3	6	67	93	46
9	8	8	37	45	68
				64	
54					

XII. (*a*) Add:

7	7	8	8	6	6	9	9
8	38, etc.;	7	57, etc.;	9	29, etc.;	6	86, etc. (oral)

(*b*) **Add at seats:**

4	7	2	6	3	34	43	34	74
3	4	4	2	2	73	74	64	44
4	3	4	3	9	34	48	47	77
7	7	7	6	4	56	95	53	23
7	4	6	4	6	67	47	69	79
6	5	3	7	4	54	76	76	41
8	7	4	3	4	13	54	85	28
5	7	8	6	3	46	57	27	34
3	7	2	4	6	64	63	86	48
4	2	8	6	9	48	46	73	3
8	6	4	3	5	97	89	24	49
7	6	3	3	9			79	83
			3	6			46	
	65							

XIII. (*a*) Add:

8	8	7	7	9	9
8	18, etc.;	9	39, etc.;	7	47, etc. (oral)

(*b*) Add at seats:

3	2	3	6	7	43	34
3	5	5	3	3	75	73
4	6	9	4	9	74	28
4	6	4	7	2	27	82
8	8	8	3	8	98	59
3	3	6	9	4	52	36
5	2	3	7	7	33	35
4	3	3	4	3	37	97
8	4	8	9	3	93	44
2	8	4	3	4	17	87
8	3	7	5	9	73	33
8	5	7	9	7	46	65

XIV. (*a*) Add:

8	8	9	9
9	49, etc.;	8	68, etc. (oral)

(*b*) Add at seats:

4	5	3	2	6
5	3	3	7	8
8	8	9	6	2
8	4	4	3	6
7	8	7	9	4
3	6	8	6	9
4	4	4	5	5
9	8	5	9	8
7	9	3	4	4
4	3	9	5	4
8	9	4	3	9
9	8	4	6	8

XV. (*a*) Add:

9	9	9	9	9
9	39	59	29	79, etc. (oral)

(*b*) Add at seats:

4	2	4	56	64	62
4	3	6	75	52	59
9	6	2	64	43	73
3	9	7	34	57	87
8	8	5	47	39	38
6	8	8	89	84	94
4	9	3	66	76	49
9	7	4	93	99	56
7	4	6	79	64	83
4	9	7	54	77	77
9	2	8	87	89	98
9	7	9	28	49	69

SECTION E

SUBTRACTION

I. Where the figures of the minuend are greater than the corresponding figures of the subtrahend.

(1) In the following examples, what must be added to the bottom line to make the top number?

$$\begin{array}{cccccccc}10 & 10 & 10 & 10 & 10 & 10 & 10 & 10\\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ \hline\end{array}$$

NOTE. The answers are to be put below the line and, when oral, given thus: for (1), "one and 9 make ten".

(2) $\begin{array}{ccc}9 & 9 & 9\\ 1 & 2 & 3,\\ \hline\end{array}$ etc.

(3) $\begin{array}{cccc}8 & 8 & 8 & 8\\ 1 & 2 & 4 & 6,\\ \hline\end{array}$ etc.

(4) Deal similarly with the other numbers as minuends.

II. In the following examples, how much greater is the top number than the bottom number?

(1) $\begin{array}{ccc}10 & 10 & 10\\ 3 & 2 & 5,\\ \hline\end{array}$ etc.

(2) $\begin{array}{cccc}9 & 9 & 9 & 9\\ 4 & 6 & 2 & 5,\\ \hline\end{array}$ etc.

Deal similarly with the other numbers as minuends. Answers, when oral, are to be given as before.

III. What is the difference between the following numbers?

(1) $\begin{array}{cccc}10 & 10 & 10 & 10\\ 4 & 7 & 3 & 2,\end{array}$ etc.

(2) Deal similarly with the other 9 digits as minuends.

IV. What is the difference between:

99	99	99	99	99	99	99	99	99
21	32	23	44	65	26	57	28	69, etc.

Answers are to be read off thus: "One and 8 make nine; two and 7 make nine".

V. When the figures of the subtrahend are not always smaller than the corresponding figures of the minuend, find the difference between the numbers:

1.	90	90	90	90	90	90	90	90
	21	32	53	64	25	36	48	69

The answers are to be read off thus: "One and 9 make ten; three and 6 make nine".

2.	91	91	91	91	91	91	91
	22	33	64	25	16	67	88

901	901	901	901	901
123	243	542	865	723
778	658	359	36	

3.	92	82	72	62	52	42	62
	23	34	65	26	27	18	19

920	902	921	912	812	721
123	234	432	656	737	138

4.	93	93	93	83	73	63	53
	24	35	25	26	38	47	19

913	923	931	903	932
124	325	446	657	868

5.	94	84	74	64	54
	25	36	37	28	39

914	924	841	842	740	644
125	316	257	375	196	128

6.						
95	85	65	95			
26	37	28	49			
945	954	953	925	915	857	
157	267	182	179	138	184	

7.						
96	86	46	66			
27	38	29	39			
946	836	726	906	961		
157	118	179	268	397		

8.						
98	78					
29	39					
908	918	981	988	958	938	983
189	128	197	199	229	369	194

SUBTRACTION PROBLEMS

(These are type questions; others similar may be made by the teacher.)

1. Johnnie had 239 butter-nuts, but the squirrels took 193. How many were left? 46.

2. A man had $420; he bought a horse for $285. How much had he left?

3. Out of a box containing 240 lemons, a fruit dealer sold 156. How many were left? 54

4. Mary bought a pair of rubbers for 65 cents. What change should she get from $1?

5. Mr. Brown's salary is $975, and his expenses are $786. How much does he save?

6. Mary's book contains 237 pages. She has read 168 pages. How many pages are left?

7. Mr. Great's farm contains 346 acres, and Mr. Small's contains 258 acres. How much larger is Mr. Great's farm than Mr. Small's?

8. A man who borrowed $685 paid back $368. How much is yet to be paid?

9. Jenny sold her sleigh for $1.25, which is 36 cents more than she paid for it. What did she pay for it?

1. Fred bought a book for 26 cents, a slate for 15 cents, and pencils for 12 cents. How much change should he get from a two-dollar bill?

2. From a farm of 425 acres, 48 acres were sold at one time, and 65 another time. How many acres remain unsold?

3. A man put 85 head of cattle into four cars. He put 18 in the first, and 21 into each of the second and third. How many were left to go into the fourth car?

4. A trader bought goods and sold them for $735, and gained by doing so $147. What did the goods cost the trader?

5. Harry weighs 76 pounds, and Frank 18 pounds less. What do the two together weigh?

6. A mile is 1,760 yards, and Tom ran 986 yards. How much less than a mile did he run?

7. Take the difference between 684 and 495 from the sum of 425 and 329.

8. John owed James $18.25. He paid him what he earned in five days, which was $11.35. How much does John now owe James?

9. Four cars carried 1,000 children to a picnic. If there were 286 in the first, 178 in the second, and 196 in the third, how many were in the fourth car?

10. In a school of 512 scholars, 286 are girls. How many more girls than boys are there in this school?

GRADED EXERCISES

G. C. D. AND L. C. M.

1. A rectangular field is 6,880 ft. long and 4,840 ft. wide. Find the length of the longest string which will measure both a side and an end of the field. 40 ft.

2. What is the smallest sum of money with which you can buy an exact number of chickens at 75c., geese at $1.25, and turkeys at $1.75? $17.75

3. What is the greatest equal length into which three trees can be cut, the first being 84 ft. long, the second 105 ft., and the third 119 ft.? 7'

4. Find the smallest number of bushels of wheat which would be equal in weight to an exact number of bushels of rye or of barley.

5. What is the length of the longest pole which will exactly measure 84 ft., 56 ft., and 98 ft.? 14'

6. What is the smallest quantity of wheat which may be taken to market in either 15, 25, 40, or 75 bushel loads?

7. A rectangular field, 1,222 ft. by 728 ft., is fenced with the longest rails possible. What is the number of rails, if the fence is straight and there is no overlapping? 150 rails

105

$112.50

8. A farmer has 100 bu. of oats, 68 bu. of corn, and 42 bu. of wheat. If he uses bags of the same size, what is the least number of bags he will have?

9. What is the smallest sum of money with which a man goes to market, if he is able therewith to buy an exact number of chickens at 50c., lambs at \$3.25, or hogs at \$9.75, and still have \$15 left?

10. What is the smallest number which will contain 9 and 21 and leave as remainder 5 in each case?

REDUCTION OF FRACTIONS

1. Write as "sixteenths" the following:

$1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, 2\frac{3}{4}, 4,$ [illegible]

2. Supply the missing numerators and denominators in:

$$\frac{3}{4}=\frac{?}{8}=\frac{?}{16}=\frac{6}{?}=\frac{12}{?}=\frac{?}{20}.$$

3. A man bought 46 half-pound packages. Into how many one-eighth pound packages can he change them? (Use fractions.)

4. Choose the smallest common denominator for fractions equivalent to: (1) [illegible] and [illegible]; (2) [illegible] and [illegible]; (3) [illegible] and [illegible].

5. Reduce the following fractions to equivalent fractions having a common denominator: (1) [illegible] and [illegible]; (2) [illegible] and [illegible]; (3) [illegible] and [illegible].

6. Reduce to fractions having the smallest common denominator: [illegible], [illegible], [illegible], $\frac{17}{20}$.

7. Reduce to equivalent fractions having a common denominator, and arrange in order of magnitude:

(1) [illegible], [illegible], $\frac{7}{8}$; (2) [illegible], [illegible], [illegible], $\frac{17}{20}$.

8. Which is the greater: (1) $4\frac{3}{5}$ or $\frac{25}{6}$? (2) $\frac{27}{5}$ or $5\frac{3}{4}$?

9. If $5\frac{1}{4}$ lb. be put into one-twentieth pound bags, how many bags will be needed?

10. How many lb. in $\frac{27}{5}$ lb. + $\frac{17}{5}$ lb. + $\frac{26}{5}$ lb. + $\frac{24}{5}$ lb.?

ADDITION AND SUBTRACTION OF FRACTIONS

1. A man broke off his whip $\frac{1}{4}$ of its length, and then $\frac{1}{3}$ of its length. What part of the whole was broken off? What part was left? How much smaller was the part left than the parts broken off?

2. A coal-miner's outfit cost as follows: drill, $\$8\frac{3}{4}$; pick, $\$\frac{7}{8}$; shovel, $\$\frac{3}{4}$; needle, $\$\frac{1}{4}$; scraper, $\$\frac{1}{4}$; axe, $\$1\frac{1}{8}$; saw, $\$\frac{7}{8}$. (1) Find the total cost. (2) Find the change from a \$20 bill.

3. A meat bill for a large hotel was as follows: $19\frac{1}{4}$ lb. lamb; $3\frac{3}{4}$ lb. cooked meat; $36\frac{7}{8}$ lb. fish; $156\frac{3}{8}$ lb. beef; $25\frac{1}{8}$ lb. pork; $28\frac{5}{8}$ lb. fowl. Find the total number of pounds.

4. In a school $\frac{25}{48}$ of the number are girls. What fraction of the number in the school do the boys form? By what fraction of the total number does the number of girls exceed the number of boys?

5. A stick is broken into two parts, so that one part is $\frac{5}{8}$ of the stick. If the difference in the parts is 2 ft., what fraction of the stick is equal to 2 ft.?

6. A boy spent $\frac{1}{3}$ of his earnings in clothes, and $\frac{3}{8}$ of them in board, and gave the remainder to his mother. What part of his earnings did she receive?

7. From the sum of $\frac{2}{3}$ and $\frac{7}{8}$, take the difference of [illegible] and $\frac{3}{8}$.

8. A man gave away $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$ of his property. How much of it had he left?

9. John's whip was $8\frac{1}{2}$ ft. long and James's was $7\frac{1}{4}$. If John breaks off $2\frac{1}{3}$ ft. from his whip, how much longer will James's whip be than John's?

10. How much smaller is the difference between $\frac{1}{3}$ and $\frac{1}{4}$ than the sum of $\frac{1}{3}$ and $\frac{1}{6}$?

FRACTIONS

1. A farmer having 2,400 bu. of potatoes sold $\frac{1}{3}$ of them at one time, $\frac{3}{8}$ at another, and 35 bu. at another. How many bushels had he left?

2. A man earns \$5 a day, but spends $\frac{4}{25}$ of his earnings for board and $\frac{3}{8}$ for clothing and other expenses. How much will he save in 4 weeks?

3. Two men 100 miles apart approach each other, each going $10\frac{5}{8}$ miles a day. After 4 days, how far apart will they be?

4. A bought a house for \$4,300 and, after spending $\frac{1}{8}$ of its cost in improvements, sold it for \$5,555. How much did he gain?

5. Mr. Brown paid $\$9\frac{3}{10}$ for cleaning his walk, $\$5\frac{5}{8}$ for trimming his vines, and $\$6\frac{3}{4}$ for cutting his lawn. He gave in payment a 20-dollar bill and a 5-dollar bill. What change should he receive?

6. The product of two fractions is $10\frac{1}{3}$ and one of them is $3\frac{5}{8}$. What is the difference between the two fractions?

7. A tank which is full of water contains 800 gal. Two pipes run out of it and one runs into it. (*a*) If one pipe can empty it in 10 hours and the other in 12 hours, how many gal. will remain in it after both these pipes have been running

4½ hours? (*b*) If the supply pipe which can fill it in 16 hours be also running, how much will remain in the tank after the 4½ hours?

8. Of my property ⅓ is in land, ¼ of the remainder in business, and ⅗ of what still remains, which is $2,400, is in the bank. How much property have I?

9. A man owning ⅜ of a vessel sold ⅘ of his share for $18,500. What is the vessel worth?

10. A owns ⅝ of a farm and B the remainder, and ⅔ of the difference of their shares is equal to $3,000. What is the farm worth?

1. If ⅝ of a ton of hay cost $16.60, what would 3¾ tons cost?

2. A man works 16½ days for $39.60. What would he earn in a year of 280 working days?

3. If 5 be added to both terms of the fraction $\frac{3}{10}$, by how much is it increased?

4. A can do a work in 6½ days which he and B can do working together in 3 days. If A gets $3.00 a day, what ought B to get?

5. A contractor employs 40 men to do a work which must be done in 16 days. After 5 days he finds it will require 20 days more. How many more men must he employ to finish the work in the 16 days?

6. A can do a work in 5 days, B in 6 days, and C in 8 days. They all work together and get $60 for the work. How should the $60 be divided?

7. Divide 100 into two parts so that ½ of the greater may be equal to ⅔ of the less.

8. A deposits in the bank $500 more than C, and ⅔ of A's deposit is equal to ¾ of C's deposit. What did each deposit?

9. The width of a room is such that ¾ of it is equal to ½ of the length of the room. If the perimeter is 170 ft., find the width.

10. For every car-load of iron dumped into a furnace, ⅞ of a car of coke was used for fuel, and ⅜ of a car of limestone was used for flux. In all 900 car-loads of ore, coke, and limestone were used a day. How many of each were used a day in the furnace?

1. A boy being asked how many fish he had, said that the difference between ½ of them and ⅔ of them is 2. How many had he?

2. Four persons owned a ship. A owns ¼ of it, B ⅓ of the remainder, C ½ of what then remains, and D's share was $3,000. What was the value of the ship?

3. $\frac{4}{21}$ of 3486 is $\frac{83}{113}$ of what number?

4. At a school examination, $\frac{5}{8}$ of the pupils passed and 125 failed. How many more were those who passed than those who failed?

5. What part of a day is 12 hr. 26 min. 40 sec.? If 4 million gallons of water pass a point in a stream in that time, how much will pass in 3 days?

6. A train goes on an average $\frac{5}{8}$ of a mile a minute. How long will it take to go 820 miles?

7. A piece of land is owned by A and B. A owns $16\frac{5}{8}$ acres and B $18\frac{3}{8}$ acres. If the whole piece is worth $3220, what is A's land worth?

8. A and B travel a journey, A going at the rate of $4\frac{1}{2}$ miles an hour and B at 5 miles an hour. A starts 2 hours before B and reaches the end 3 hours after B. (*a*) Where did B pass A? (*b*) How long was the journey?

9. The product of two numbers is $2\frac{1}{11}$ and one of the numbers is $\frac{3}{22}$; what is the other?

10. If a boy 4 ft. high increases his height by $\frac{1}{16}$ of his height every year for 4 years, how much taller is he at the end of 4 years?

MEASUREMENT

1. A building lot contains $\frac{1}{2}$ acre and has a frontage of 60 ft. Find its depth.

2. How many sods 10 in. by 12 in. will be required to turf a lawn 100 ft. long and 50 ft. 6 in. wide?

3 How many yards of carpet $\frac{3}{4}$ yd. wide will be required for a floor 20 ft. long and 17 ft. wide, strips running across the room?

4. What will it cost to paper the walls of a room 18 ft. long, 12 ft. wide, and 9 ft. high, with paper 8 yd. to the roll and $\frac{1}{2}$ yd. wide, at 45 cents a roll?

5. Find the cost of plastering the walls and ceiling of a room 36 ft. long, 27 ft. wide, and 9 ft. high, at 25c. per sq. yd.

6. A close fence 6 ft. high surrounds a vacant lot 600 ft. by 380 ft. At 8c. a sq. yd., what will it cost to paint both sides of the fence?

7. Find the cost of paving 810 ft. of street 60 ft. wide, if the concrete foundation cost 65c. a sq. yd., and the asphalt surface $1.25 a sq. yd.

8. Find the cost of a 6-ft. side-walk around a block 30 yd. by 40 yd. at $1.15 a sq. yd.

9. A square, open bowling-green 60 ft. on a side is sodded with sods 15 in. by 9 in. How many will it take?

10. A paper machine turns out a strip of paper 500 ft. long and 120 inches wide each minute. How many sq. yd. of paper does it turn out in an hour?

BOARD MEASURE

1. How much lumber will be required for an open board fence 4 boards high with boards 8 in. wide and 6 in. apart?

2. Find the cost of 426 planks 4 ft. 8 in. long, 1 ft. wide, and 2 in. thick, at $26 a thousand feet.

3. A stick of square timber 36 ft. long and 9 in. by 9 in. at the end, is bought at $12 per M. Find its cost.

4. A box without a cover is made of 2 in. lumber. How many board ft. are required, if its inside dimensions are 2 ft. 8 in. long, 2 ft. wide, and 1 ft. 6 in. deep?

5. At $24 per M, what will it cost for the flooring of a room 21 ft. long and 16 ft. wide, if ¼ of the lumber is lost in matching?

6. A side-walk 30 rods long and 6 ft. wide is made of 2 in. plank supported on three rows of scantlings 3 in. by 4 in. Find its cost at $18 per M.

7. A bin 9 ft. square and 6 ft. high is made of 2 in. lumber nailed to uprights 4 in. by 4 in. standing 3 ft. apart, the bottom resting on timbers 4 in. by 6 in. and 3 ft. apart. What is the cost of the lumber at $16 per M?

8. A barn 100 ft. long and 60 ft. wide is 30 ft. high to the gables. If the gables are together equal to ⅓ of a side, what will it cost to inclose the barn with inch lumber at $12 per M?

9. A covered box whose outside dimensions are 5 ft., 6 ft., and 8 ft. is made of 2 in. lumber. Find the cost of the lumber at $22 per M.

10. Eighteen joists 10 in. by 12 in. and 16 ft. long, are made by spiking together 2 in. planks bought at $16 a thousand. What did the joists cost?

LENGTHS AND SURFACES

1. A rectangle is 6 in. x 8 in. What is the length of its diagonal?

2. A rectangle is 40 ft. long and its diagonal is 50 ft. What is its width? Its area?

3. Two vessels start at the same time from A, one going due north at 12 miles an hour, and the other due west at 16 miles an hour. How far will they be apart in 4 hours?

4. How far has the first vessel gone when the vessels are 80 miles apart?

5. A ladder stands against a wall; if the ladder be 40 ft. long and its foot 6 ft. from the wall, how high on the wall does it reach?

6. How far apart are two boys who stand at opposite corners of a 10-acre square field?

7. A school-room floor is 30 ft. by 40 ft. and the height of the ceiling is 12 ft. How far is it from the upper corner to the opposite lower corner of the room?

8. A cube with a 12-in. edge has a silver wire running from one corner through the centre to the opposite corner. How long is the wire?

9. A gable is 60 ft. wide and 40 ft. high. How long is the roof from the highest point to the eaves?

10. The three sides of a right-angled isosceles triangle together measure 3414 ft. How long is one of the equal sides?

VOLUMES

1. What is the weight of a load of ice of 30 blocks 24 in. x 30 in. and 1 in. thick, if a cu. ft. of ice weighs 56¼ lb.?

2. A box 12 ft. by 8 ft. has water in it to a depth of 3½ ft. How many gallons are there if a cu. ft. contains 6¼ gallons?

3. A school-room 30 ft. long, 24 ft. wide, and 12 ft. high, seats 50 pupils. How many cu. ft. of air are there for each pupil?

4. A hothouse bed 5 ft. long and 3 ft. 4 in. wide contains 6 cu. yd. of earth, with 6 in. of space above it. What is the depth of the hotbed?

5. How many cubes of 2½ in. edge can be sawed from a block 10 ft. 2½ in. long, 6 ft. 5 in. deep, and 6 ft. 8 in. thick?

6. A box 6 ft. by 8 ft. contains 240 cu. ft. of water. How deep is the water?

7. A ditch ½ mile long is 3 ft. wide and 18 in. deep. How many cu. yd. of earth were removed in digging it?

8. How high must a pile of wood be if it is 8 ft. wide and 50 ft. long, and contains 30 cords?

9. A shed is 24 ft. long, 18 ft. wide, and 9 ft. high. How many cords of wood will it hold?

10. A bushel fills 2150 cu. in. of space. How many bushels of grain can be put into a bin 4 ft. wide, 6 ft. long, and 5 ft. high?

SURFACES AND VOLUMES

1. A square field contains 10 acres. What is its perimeter?

2. A lot is 144 ft. long and 64 ft. wide. Find the side of the square lot that has the same area.

3. How much greater is the perimeter of the first lot than that of the second named in 2?

4. A field contains 20 acres and its length is twice its breadth. Find its perimeter.

5. The length of a field is 1½ times its breadth. If its area is 15 acres, what is its length?

6. Find the length in rods of the side of a square whose area is 4 acres.

7. A square field contains 40 acres. How many more rods of fence is needed to fence the 40 acres when the length is 4 times the breadth?

8. A rectangular solid 20 ft. long with a square end, contains 980 cu. ft. Find the end dimensions.

9. The end of a rectangular solid is twice as long as it is wide. If its volume is 384 cu. in., what are the end dimensions?

10. A ditch 1½ times as wide as it is deep is ½ a mile long and contains 62360 cu. ft. of water when full. What is its depth?

CIRCLES—LENGTHS

1. Find the length of the tire of a 5-ft. wheel.

2. The circumference of a wheel is 11 ft. Find its radius.

3. A circular race-track is 1 mile in circumference. Find its diameter in rods.

4. The inner side of a bicycle track is ½ a mile. If the track is 2 rods broad, how long is the outer part of it?

5. A wheel turns 1600 times in going 3½ miles. Find its radius.

6. If the front wheel of a carriage is 3 ft. 8 in. in diameter and the hind wheel 4 ft. 2 in., how many revolutions does the front wheel make more than the hind wheel, in going a mile?

7. The radius of a fountain is 21 ft. Find the cost of fencing the fountain with an iron railing at $4.20 a yard.

8. The diagonal of a square is 7 ft. Find the length of the circumference of the circle which circumscribes the square.

9. A rectangle 40 ft. long by 30 wide has semicircular ends put on it. Find the perimeter of the whole figure.

10. A spot on a belt passing over a wheel 1¾ ft. in radius is seen to move 10 ft. in 2 sec. How many revolutions does the wheel make a minute?

LENGTHS AND SURFACES—CIRCLES

1. If a circular track has a radius of 35 ft., what area is inclosed by it?

2. If a cow be tethered by a 70-ft. rope, over what part of an acre can she graze?

3. A large circle with radius 7 in. has a smaller circle with radius 5 in. placed centrally on it. Find the difference in area between the 2 in. ring and the inner circle.

4. What is the difference in rods in the perimeter of two fields, each of which contains 10 ac., but where one is circular and the other square?

5. Find the number of sq. in. of tin required to make a cylindrical can with cover, if the diameter be 4⅜ in. and height 6 in.

6. A roller is 12 ft. long and 2⅓ ft. in diameter. Find its entire surface.

7. If it cost $63.36 to paint a cylindrical pillar 21 ft. high at 36 cents a sq. ft., find the diameter of the pillar.

8. How often would the roller of Question 6 turn in rolling a square 10-acre field?

9. A smoke pipe is 7 inches in diameter. Find the area of its cross section and the diameter of a pipe which would let 4 times the smoke through.

10. A cent is 1 in. in diameter. What should be the diameter of a 2-cent coin of the same material and thickness?

SURFACES—TRIANGLES AND CONES

1. A rectangle is 6 ft. by 4 ft. Find its area and the area of a triangle whose base is 6 ft. and height 4 ft.

2. Five triangles have bases 3½ ft., 5½ ft., 4¾ ft., 6¼ ft., and 8 ft. respectively, and the height of each is 10 ft. Find the sum of their areas in the shortest way.

3. A circle with radius 14 inches has a sector whose arc is 22 inches cut out of it. Find the area of the sector by considering it as made up of a number of small triangles whose bases form the arc.

4. The sector of Question 3 is made into a hollow cone. Find its area. Find also its perpendicular height.

5. If the cone in Question 4 were solid, what would be its entire surface?

6. If the diameter of the base of a cone be 12 in. and its height 8 in., find its entire surface.

7. A silo is cylindrical in shape, with a conical roof. If the diameter be 14 ft., the total height 29⅓ ft., and the height to the roof 20 ft., find the entire outer surface.

8. The slant height of a cone is 10 ft., and the circumference of the base 15 ft. Find the area of the curved surface.

9. A cone is 3½ ft. in diameter and its perpendicular height is 2⅓ ft. Find its curved surface.

10. How high must a cone be whose circumference is 44 ft., in order that its curved surface may contain 256⅔ sq. ft.?

VOLUME—CAPACITY

1. A cylinder has a base containing 154 sq. in. If it is 20 in. high, find its volume.

2. A cone is ⅓ the volume of the corresponding cylinder. Find the volume of a cone which is 10 ft. high and has a diameter measuring 7 in.

3. Find the weight of 6 ft. of water in a cistern whose diameter is 14 ft. (A cu. ft. of water weighs 62½ lb.)

4. What is the cubic content of a cylindrical silo whose diameter is 14 ft. and whose height is 20 ft., if it have upon it a conical roof 12 ft. high?

5. How many barrels of water are there in a cylindrical tank 10½ ft. in diameter, if it be filled to a depth of 12 feet, and a cu. ft. contains 6¼ gallons? (A barrel of water is 31½ gal.)

6. A pile of coal in the shape of a cone is 30 ft. high, and 154 ft. in circumference at its base. Find its volume.

7. A bushel contains 2,218 cu. in. Find how high the wheat must be in a bin 9 ft. by 8 ft. to contain 600 bu.

8. How deep in a cylinder whose diameter is 7 ft. must water stand in order that there may be 60 barrels of water? (1 bbl.=31½ gal.; 1 gal. weighs 10 lb.)

9. A hollow tube made of lead is 100 ft. long, its inner diameter is 2¼ in., and its outer 2½ in. Find its weight, if lead weighs 11.38 times as much as water.

10. Find the weight of a solid cylinder of lead whose diameter is 7 in. and length 30 in.

MISCELLANEOUS PROBLEMS

1. Find the total daily and weekly sales for each department in the following record of sales of a department store:

Departments	Mon.		Tues.		Wed.		Thur.		Fri.		Sat.		Total for week
	$	c.	$	c.	$	c.	$	c.	$	c.	$	c.	
Dress Goods....	123	42	135	23	213	40	119	20	216	45	225	19	$
Hosiery........	26	98	91	22	82	26	23	65	29	02	225	14	$
Gloves.........	121	10	73	44	125	90	210	14	136	45	136	44	$
Jewelry........	113	95	83	65	134	38	256	15	124	25	542	27	$
Toys	63	22	3	22	124	27	28	12	62	10	22	15	$
Books	120	00	214	94	28	15	334	15	343	15	219	05	$
Carpets........	544	00	353	62	262	20	167	13	446	45	644	00	$
Art Department	115	20	128	18	139	40	125	15	125	16	122	56	$
Notions........	223	05	134	15	405	25	393	42	496	80	116	25	$
Crockery	46	72	269	13	150	00	118	24	13	95	42	24	$
Totals........	$		$		$		$		$		$		$

2. A messenger boy buys peaches at the rate of 4 peaches for 5 cents. How many dozen must he sell daily at the rate of 7 peaches for 10 cents in order that he may make $1.50 a day?

3. Three men working at the same rate a day did a piece of work for $90. The first worked 5 days, the second 6⅔ days, and the third 8⅓ days. How much did each receive?

4. A baker made 252 lb. of vanilla-cream biscuit and sold it at 18c. a pound. He used the following ingredients: 1 bbl. of flour at $4.75; 20 lb. butter at 25c.; 16 lb. lard at 12½c.; 64 lb. powdered sugar at 6¼c.; 3½ gal. milk at 6c. a qt.; 2 qt. glycerine at 75c.; 10 oz. soda at 8c. a lb.; salt, ammonia, and vanilla, $1.05; 2 gal. eggs (10 eggs to a pint) at 27c. a dozen. Find his gain and gain per cent.

5. Separate 249984 into prime factors, and show that it is the continued product of three consecutive numbers.

6. Find, with as little work as possible, the value of

(a) $2\frac{7}{12}+4\frac{1}{6}+\frac{7}{8}+1\frac{3}{4}+\frac{11}{18}$; (b) $\frac{6}{11}-4\frac{1}{8}-2\frac{7}{33}+3\frac{25}{88}-\frac{1}{12}$.

7. A cargo worth $16,300 was insured at 3¾% for 90% of its value. Find the actual loss to the owner in case of shipwreck.

8. A train leaves a station X at 9.30 a.m. Tuesday, and arrives at Y at 8.20 p.m. Wednesday. At what rate is the train travelling if the distance from X to Y is 1,045 miles?

9. Find to the nearest cent the sum which must be deposited in a savings' bank on January 31st, so that with interest at 3% per annum the amount will be sufficient to pay a debt of $500 due on July 1st.

10. At $6.75 a ton estimate the cost of coal required to fill a bin 22 ft. long, 12 ft. wide, and 6 ft. deep, allowing 35 cu. ft. to the ton.

11. Complete the following pay roll:

Name	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Total Hours	Rate	Amount
								Ct.	
Edwin Irving ...	8½	8	8¼	8	8½	8		30	
Allen Gray	8	8¼	8	8½	8	8		25	
Jack O'Neil.....	8¼	8	8½	8	7	8		20	
Henry Ward	8½	8	8¼	8	8	6		25	
John Young.....	8	8	.....	5	9	8		22	
William Colby ..	8½	8	7¼	8	8	8		20	
James Fitz......	8	7	7½	8	8	7		18	
Fred Clark......	6	8	6½	8	8	8		12½	
Jack Mason.....	8	8	8	8	7½	8		12½	
Edwin Gale.....	8	7	8	8	6	8		30	
Robert Gale.....	8	7	8	8	6	8		25	
Gilbert Beecher .	8	8	.....	7	8	8		15	
Allen Files	9	8½	8	8	7	7½		12½	
Frank Colby....	7	8	8½	3	6½	8		12	
Fred Brooks....	8	8	8	8	8	8		37½	
George Foote....	6½	8	5	8	9	5½		27½	
Total.........	....	....	....	....	....	....	....	....	....

12. A merchant bought 522 eggs at 28 cents a dozen, and the same number at 34 cents a dozen. He found 7 dozen of them damaged, and sold the rest at 39 cents a dozen. Find his gain.

13. What will 4 bu. 2 pk. 3 qt. 1 pt. of nuts cost at $2.25 a bushel?

14. (*a*) What are the prime factors of 32, 320?
   (*b*) Find the greatest common measure of 372, 474, and 582.
   (*c*) Find the least common multiple of 35, 63, 91, and 119.

15. If a man can row 4⅔ miles an hour in still water, how many miles can he row in 3½ hours up a river that flows at the rate of 1⅞ miles an hour?

16. A man fails owing debts amounting to $10,500. If his property is worth $4,650, how many cents on the dollar can he pay his creditors?

17. Make out a receipted bill for the following items: May 6, 1905. Mr. R. C. Stair bought of Jones Bros., 14 yd. silk at $2.25; May 10, 68 yd. cotton at 8½c.; May 12, 15 yd. tweed at $2.75; May 17, 24 yd. carpet at $1.87½; June 5, 54 yd. matting at 37½c.; June 12, six pairs curtains at $3.50.

18. An agent bought some flour, paid $54.50 storage, and charged $180 commission. His entire bill was $8,234.50 What was the rate of commission?

19. A person borrows $2,560, and, at the end of each year, pays $650 to reduce the principal, and to pay interest at the rate of 5 per cent. on the sum which has been standing against him through the year. How much will remain of the debt at the end of three years?

20. What is the acreage of a rectangular field whose length is 234 rd., and whose breadth is 165 rd.?

21. Complete the division below by writing in the missing numbers in the first and second lines:

$$\begin{array}{r} 9\underline{)\qquad\qquad\qquad\qquad\qquad\qquad} \\ 7\underline{)\qquad\qquad\qquad;\text{ remainder } 8} \\ 2{,}768;\text{ remainder } 5 \end{array}$$

22. (*a*) Find the H.C.F. and L.C.M. of 7,648, 13,384, and 63,096, by using their prime factors.
   (*b*) Find the value of $\cdot 0256 \times 1\cdot 0071 \div 2\cdot 7975$.

23. How long will it take a train, running at the rate of 21¼ miles an hour, to travel 17 miles?

24. In 1904 a steamship company owned 26 steamers with an aggregate tonnage of 126,185 tons. Find the average tonnage of the company's steamers in that year. In 1905, two

additional steamers were added, each with a tonnage of 10,754 tons. By how much did the additional tonnage increase the average? In each case give the answer to the nearest ton.

25. (*a*) A note for $300 drawn on March 1st, 1911, for four months without interest, is discounted by a bank on April 2nd, 1911, at 6%. Find the proceeds.
(*b*) If the note carried interest at 5% per annum, what would the proceeds be?

26. A house worth $2,400 was insured for three fourths of its value, at a yearly premium of 2 per cent. During the third year the house was burned. Find the owner's net loss, and also the net loss of the insurance company.

27. A bankrupt's assets are found to be ⅔ of his liabilities, but on examination ¼ of his assets prove to be worth only 50 cents on the dollar. How many cents on the dollar can he pay to his creditors?

28. Find the total number of board feet of lumber in the following:

5 pieces 2 in. × 6 in. × 12 ft.
37 pieces 4 in. × 4 in. × 12 ft.
15 pieces 2 in. × 4 in. × 10 ft.
9 pieces 3 in. × 3 in. × 15 ft.

29. A commission merchant sold 4,500 dozen oranges at 32c. a dozen. After deducting $27.40 for freight, $15 for storage, and his commission, he returned $1,340 to his employer. What was the rate of his commission?

30. If a bundle of laths cover 6 sq. yd., find the cost of lathing and plastering both sides of a partition 36 ft. long and 18 ft. high, the laths costing 25c. a bundle and the plastering 18c. a square yard.

31. (*a*) Find the value of 91.512 × .576 ÷ 3.72.
(*b*) Find the value of (4½ + 3⅝ — 2⅓) ÷ (7⅓ — 3¾ + 2).

32. Explain the following method for dividing ⅚ by ⅔:

$$\begin{array}{c|c} 2 & 5 \\ \hline 3 & 6 \\ \hline 4 & 5 = 1\tfrac{1}{4} \end{array}$$

33. A man keeps ⅗ of his farm in pasture, ¾ of the remainder under cultivation, and the rest, 65 acres, in forest. How many acres are there in the whole farm, and what is the average cost an acre if the whole is valued at $21,775?

34. Multiply 3 mi. 59 rd. 3 yd. 2 ft. 6 in. by 7, and divide the product by 4.

35. Find the size of the largest square slab which could be used to pave a courtyard 72 ft. long and 21 yd. wide.

36. The property of a town is assessed for \$1,061,500, and that of a certain resident for \$4,500. What special tax must this resident pay toward the building of a school-house costing \$5,500?

37. Which is more, and how much, a discount of 40 and 10 off, or a discount of 20 and 30 off, from a bill of \$3,350?

38. If a sovereign weighs 5 dwt. 8 grs., what is the value, in Canadian money, of 16 lb. 3 oz. 4 dwt. of the same metal, the sovereign being equivalent to \$4.86$\frac{2}{3}$?

39. A note for \$300 was given April 12th, 1906, bearing interest at 7%. On this note the following sums were paid: Jan. 1st, 1907, \$75; July 1st, 1907, \$80; Sept. 8th, 1907, \$125. Find how much was due Jan. 1st, 1908.

40. Allowing 6$\frac{1}{4}$ gallons to the cubic foot and 31$\frac{1}{2}$ gallons to the barrel, how many barrels of water will it take to fill a rectangular cistern 5 ft. long, 4$\frac{1}{2}$ ft. wide, and 15 ft. deep?

41. Simplify the following complex fractions:

$$(a)\ \frac{28-\frac{1}{3}}{\frac{3}{4}\times\frac{4}{9}};\qquad (b)\ \frac{4\frac{1}{2}-2\frac{3}{4}}{8\div2\frac{3}{4}};\qquad (c)\ \frac{1+\frac{3}{4}\text{ of }\frac{8}{15}}{2\frac{1}{2}-\frac{5}{8}}.$$

42. (*a*) Change 3 pk. 4 qt. to the fraction of a bushel.

(*b*) Find the cost of 8 yd. 1 ft. 6 in. of pipe, 4 lb. to the foot, at 25c. a pound.

(*c*) What part of $\frac{1}{18}$ of 5$\frac{1}{2}$ is $\frac{3}{4}$ of $\frac{11}{12}$?

43. A vessel A contains 21 gallons of wine and 9 gallons of water, and a second vessel B contains 24 gallons of wine and 6 gallons of water. If a third vessel D is filled by taking 7$\frac{1}{2}$ gallons from A and 22$\frac{1}{2}$ gallons from B, how many gallons of wine and how many gallons of water will there be in D?

44. During the last four months of the year a man's average daily expenditure was \$1.38. If his average was \$1.56 for September, 97 cents for October, and \$1.24 for November, find his average, to the nearest cent, for December.

45. A contractor undertakes to build a piece of road in 60 days. He begins the work with 35 men, but finds that he has completed only $\frac{5}{8}$ of it in 40 days. How many additional men must he now engage in order to finish his contract on time?

46. Make out, in proper form, a bill for the following: 1,225 ft. spruce @ \$23.50; 1,250 ft. pine @ \$40; 1,890 ft. maple @ \$47; 1,318 ft. hemlock @ \$42.80; 390 ft. basswood @ \$31; 1,530 ft. oak @ \$75.50; 1,620 ft. maple @ \$47; 2,325 ft. chestnut @ \$32.25 per M.

47. A Toronto merchant imported goods invoiced in New York at \$3.60 a yard. He paid an ad valorem duty of 12$\frac{1}{2}$% and marked the goods for sale at such a price as would permit him to give a discount of 16$\frac{2}{3}$%, and still leave him a clear profit of 33$\frac{1}{3}$%. Find the marked price a yard.

48. A farmer sold 350 bushels of wheat at 80c. a bushel, and received for it a 60-day note which he immediately discounted at the bank at 6%. What were the proceeds?

49. What will be the cost of shingles, at \$3.75 a thousand, to cover the two sides of a roof, each side measuring 32 ft. long and 17½ ft. from eaves to peak, if the shingles are laid 5 inches to the weather, and if $\frac{1}{20}$ of the shingles bought are wasted?

50. Assuming that the circumference of a circle is 3·1416 times the diameter, find the length of steel wire which, evenly laid, goes 40 times round a winding drum 15 ft. in diameter.

51. Coffee costing 35c. a lb. is mixed with chicory worth 10c. a lb. in the proportion of 5 lb. of coffee to 2 lb. of chicory, and the mixture is sold for 34c. a lb. Find the gain per cent.

52. A man bought cordwood for \$140, and by selling it at \$5.70 per cord, he gained \$59.50 on the lot. How many cords did he sell?

53. A man bought 43 oxen, paying as many dollars for each ox as there were oxen in the drove. He sold 20 of them at \$40 each and the rest at \$48 each. Did he gain or lose, and how much?

54. The new Canadian ten-dollar gold piece will weigh 258 grains, and will be $\frac{9}{10}$ fine, that is $\frac{9}{10}$ of the coin will be pure gold and $\frac{1}{10}$ alloy. Find in oz. (Troy) the amount of alloy required for 100,000 coins.

55. 1,869 sovereigns weigh 40 lb. Troy and are $\frac{11}{12}$ fine; find in grains the weight of pure gold in one sovereign.

56. Use the figures given in the last two problems to find how many times as great as the weight of pure gold in one sovereign is the weight of pure gold in a ten-dollar gold piece.

57. A grocer bought 2,400 lb. of sugar at 3⅞c. a lb. If 44 lb. are wasted in handling, at what rate (lb. to the dollar) must he sell the remainder so as to make a profit of 33⅓%?

58. A merchant bought two casks of wine, each containing 41 gal. 3 qt., at \$1.80 a gallon. One seventh of it leaked away. He sold 9 kegs, each containing 5 gal. 1 qt. at the rate of 30c. a pint, and the remainder at the rate of 40c. a pint. Find his gain.

59. How many pencils 7 in. long can be made from a block of red cedar 7 in. by 10½ in. by 2¼ in., if the block is sawed into strips 3½ in. wide and $\frac{3}{16}$ in. thick, each strip making the halves of 6 pencils?

60. A real estate agent sold a house for \$5,400 and charged a commission at the rate of 2½%. What was the amount of the commission?

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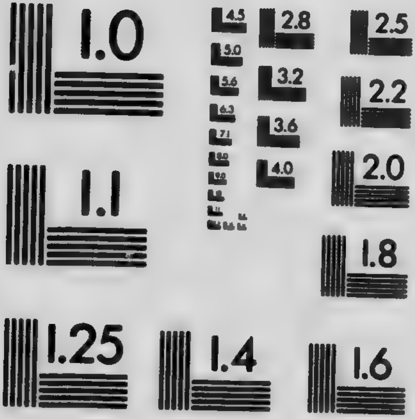

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61. A real estate agent charged a commission at a rate of 3% for selling property. He charged $240 for selling a house; what was the net amount received by the owner out of the sale?

62. The owner of a farm received $2,450 net out of the proceeds of the sale of his farm after the agent had deducted a commission of 2% of the selling price. Find amount of agent's commission.

63. A real estate agent keeps $200 out of the proceeds of a sale of property in payment of his commission, and sends the owner $7,800. What rate was his commission?

64. A citizen buys a city lot 50 ft. wide at $75 a foot frontage, gives a contractor $7,500 for erecting a house, and pays an architect 5% of the cost of the house for drawing plans and supervising building. Find total cost of the house and lot.

65. A side-walk is 440 yards long and 8 feet wide, and is made of plank 2 inches thick. The planks rest on three continuous lines of scantling 3 inches by 4 inches. Find the cost of the lumber at $15.00 per M.

66. The sovereign is the British gold coin; 1,869 sovereigns weigh 40 lb. Troy and are $\frac{11}{12}$ fine, that is $\frac{11}{12}$ of the metal is pure gold. A new gold coin, a ten-dollar piece, will be coined shortly in Canada; it will weigh 258 grains and will be $\frac{9}{10}$ fine. By comparing the amount of pure gold in these coins, find the value in dollars and cents of a British sovereign.

67. The tax rate in Toronto for the year 1910 was $11\frac{7}{10}$ mills on the dollar for general purposes, and $5\frac{8}{10}$ on the dollar for school purposes. Find the total amount of taxes paid on a property assessed at $5,400.

68. What amount of taxes will a man pay on an income of $1,800 if $1,000 are exempt, and the tax rate is 18 mills on the dollar?

69. A buyer buys oranges at the rate of $1.12 a box of 12 dozen and sells at the rate of 3 oranges for 5c. If 6 oranges in each box are wasted, find his gain.

70. In the year 1904 there were 19,431 miles of steam railway in Canada, in 1908 there were 22,966 miles; what rate per cent. was the increase during the four years?

71. Canada had 766 miles of electric railway in 1904, and 992 miles in 1908. What was the rate of increase during that period?

72. The C. P. R. is 10,564 miles in length, the G. T. R. 3,570 miles. Give an approximate comparison of their lengths in simple numbers.

73. Find the bank discount and proceeds of a three months' note for $500, bearing interest at 6% per annum, dated Jan. 5th, 1907, and discounted August 5th, 1907, at 6%.

74. The volume of an Imperial gallon is 277·274 cubic inches, that of the wine gallon used in the United States is 231 cubic inches. State approximately their relative sizes in simple figures.

75. A man's income is $1,650 a year, he spends $693; what per cent. of his income does he save? What per cent. of the amount which he spends is the amount he saves?

76. A starts 3 min. after B for a place 4½ miles distant. B on reaching his destination immediately returns, and after walking 1 mile meets A. If A's speed is 1 mile in 18 min., what is B's speed?

77. The population of a city is 340,500. Of this number 40,000 are of school age; what per cent. of the total population are the children of school age?

78. Canada exported merchandise valued at $147,748,085 to the British Empire in 1908; merchandise valued at $113,520,500 to the United States; and merchandise valued at $18,738,021 to other foreign countries. What percentage of the total exports were sent to the British Empire, and what percentage to the United States?

79. If a man receives a dividend of 3% on one fourth of his capital, a dividend of 5% on two thirds of it, and a dividend of 11% on the remainder, what rate does he receive on the whole?

80. A drover bought 50 bullocks whose total weight was 52,350 lb., at 5c. a lb. He kept them for two months at a cost of $1.50 each a month. During the time 35 increased in weight on an average of 200 lb. each, 13 increased on an average of 150 lb. each, one lost 200 lb., and one was injured at the end of the first month and was sold for $55. He weighed 1,200 lb. at time of purchase. The one that lost in weight was sold at 5c. a lb., his weight at time of purchase was 1,050 lb. The others were sold at $6.15 per cwt. Find the drover's gain or loss.

81. In a certain school 6 pupils are enrolled during the year. They attend 195, 190, 200, 201, 208 days respectively during the year. There are 210 teaching days in the year. Find (1) the average daily attendance at the school; (2) the average number of days the pupils attended during the year.

82. A grocer bought 760 lb. of sugar at 7½c. per lb., 840 lb. at 6¾c. per lb., 960 lb. at 7⅛c. per lb. (1) Find the average cost per lb. (2) Find the average number of lb. bought for one dollar.

83. A workman receives $1.75 a day for 30 days' work, $2 a day for 60 days' work, and $2.50 a day for 120 days. Find his average daily wages.

84. A grocer bought 7,960 lb. of sugar at a rate of 4⅛c. per lb., and sold at a rate of 20 lb. for one dollar. If 10 lb. were wasted in handling, find the grocer's rate of profit.

85. A jeweller requires 120 ounces of an alloy made up of 3 parts by weight of gold, to one of silver. The only gold he can get is in an alloy made up of 5 parts gold to one of silver. How much of this alloy must he use, and how much silver must he add to it to make up the alloy required?

86. A Toronto coal dealer buys coal at the mine at the rate of $3.75 for 2,240 lb., pays 25c. a cwt. for freight, and 10c. a cwt. for delivery. If he sells at $6.50 a ton, find his rate of gain.

87. The taxable value of a property in a town is $765,000, and the rate of taxation $·007 on the dollar. Find the amount of taxes.

88. Mr. Henry bought, through a real estate agent, a house for $5,000, and paid the agent a 2% commission. He insures the house at ½% for ⅔ its actual cost, and pays annually $173.50 for taxes and repairs. If he rents the house at $35 per month, what yearly rate of interest will he receive on the money invested?

89. One merchant offers to sell neckties at $6.25 a dozen with discounts of 20% and 10%; another offers the same grade at $7.00 a dozen with discounts of 15% and 16⅔%. Which is the better offer?

90. An agent sold 850 barrels of flour and received a commission of $114.75 therefor; the rate of commission was 2½%; find the selling price of the flour a barrel.

91. A share in a certain bank yields a yearly dividend of $10. If it cost $218, find to two decimal places the rate of interest the investor makes on his investment.

92. A grocer sells tea at 55c. a lb., gaining thereby 10%. What would have been his loss per cent. had he sold it at 45c. a lb.?

93. A man pays for insurance on his life 15% of his gross income; and after paying an income tax on the remainder at 6d. in the pound, he had £513 16s. 6d. left. What was his gross income?

94. An apartment house containing 6 suites of apartments cost $18,000. Three suites rent for $30 a month, three at $35 a month. Taxes are at the rate of 17 mills on the dollar on an assessment of $16,000; repairs cost $88 a year. Find the owner's net proceeds for a year.

95. There is deposited in a savings' bank $1; interest at the rate of 4% per annum is added to the principal at the end of each year. Find the amount at the end of 4 years.

96. Compile a table showing the interest on $100 for 10 days, 11 days, 12 days, 13 days, 14 days, 15 days, respectively, at 6% per annum.

97. Compile a table which will show the interest on $1 for 10 days, 20 days, 30 days, 40 days, 50 days, respectively, at 6% per annum.

98. Use the table compiled in the preceding problem to find the interest on $27 for 30 days at 6%.

99. Use the table compiled in a previous problem to find the interest on $35 for 50 days at 6%.

100. A depositor in a savings' bank makes the following deposits and withdrawals, namely: July 5th, deposit $800; July 28th, deposit $160; August 15th, withdrawal $400; Sept. 4th, withdrawal $200; Sept. 29th, deposit $750; Oct. 2nd, deposit $800; Oct. 27th, withdrawal $900; Nov. 12th, withdrawal $150; Dec. 14th, deposit $500. If the bank pays interest at the rate of 4 per cent. per annum on minimum monthly balances, with how much interest will the depositor be credited on December 31st?

101. At the beginning of a year a merchant purchased goods for which he paid $700. He sold them out completely in six months at a profit of 8%; then took the proceeds and bought a new stock of goods; he sold them out completely in the next six months at a profit of 8%. Find his profits for the year.

102. Find the interest on $640 at 5½% from May 1st to Sept. 14th.

103. Find the interest on $86.50 at 6% from Jan. 15th, 1896, to October 20th, 1896.

104. A farmer bought 25 head of cattle of an average weight of 700 lb. each at 4½c. a lb., and kept them a year at a cost of $1 a month each. He then sold them at $5.75 per cwt. If they averaged 1,200 lb. each in weight, find the selling price and what fraction of the original cost his gain or loss was.

105. A baseball team won 67 games and lost 34 during a season. What fraction of the total number played was the number won? Express the answer as a decimal fraction correct to four places.

106. The population of Algoma District in 1890 was 4,926; in 1909 it was 23,059. What was the rate of increase for the period?

107. The population of Huron County in 1890 was 61,777; in [illegible]09 it was 59,934. By what per cent. did the population decrease during the period?

108. The population of a city was 375,460. The number of children born in it during a certain year was 5,600. What was the birth rate per thousand of population? Give answer correct to two decimal places.

109. A retail merchant bought goods invoiced at $742.50, but subject to a discount of 20 and 16⅔% off. He sold them at an advance of 33⅓% on their actual cost. Find his net gain if 5% of the sales was not collectable.

110. Find the duty on 20 doz. books invoiced at 27c. each, rate of duty 10% ad valorem.

111. An importer of tweeds paid $240 in customs duties on a consignment of tweeds valued at $1,200. What was the rate of duty?

112. A druggist buys a gross of bottles of a patent medicine at $70.00; 4 bottles were broken in shipment, 4 were given away, and the remainder sold at a rate of $1 a bottle. Find his gain per cent.

113. A merchant's rate of gain on a certain line of goods is 12½%, the amount of gain $175; find the cost of the goods.

114. The population of Ontario in 1907 was 2,199,563; in 1908 it was 2,344,385. Find the rate of increase.

115. The total assessment of rural municipalities in Ontario for 1909 was $607,173,285, and the total taxes imposed $7,149,315. Find the rate in mills on the dollar.

116. The total assessment of 18 cities in the Province of Ontario for the year 1909 was $453,311,559, and the total taxes imposed were $10,535,285. Find the rate in mills on the dollar.

117. A note for $1,000 dated Sept. 1st, 1905, at 6% interest had the following payments endorsed on it: Nov. 8th, 1905, $200; Dec. 12th, 1905, $200; Feb. 3rd, 1906, $200; May 1st, 1906, $200. How much was due Sept. 1st, 1906?

118. A savings' bank pays interest at the rate of 3% per annum on minimum monthly balances A depositor made deposits and withdrawals as follows: Jan. 20th,, deposited $140; Jan. 28th, deposited $60; Feb. 14th, withdrew $20; March 16th, withdrew $20; April 20th, deposited $150; May 1st, deposited $150; May 10th, deposited $160; July 4th, withdrew $50; Sept. 10th, withdrew $20; Nov. 10th, withdrew $20; Nov. 10th, deposited $200; Dec. 15th, withdrew $50. What amount of interest was placed to his credit on Dec. 31st?

119. A deposits $1,000 in a savings' bank that pays interest at the rate of 4% per annum, payable half yearly, that is, the interest is paid at the end of each half year and added to the principal; B deposits $1,000 in a savings' bank that pays interest at the rate of 4% per annum, payable yearly, that is, the interest is paid at the end of each year and added to the principal. How much better off will A be at the end of one year than B?

120. Under the conditions of the previous problem how much better off will A be at the end of two years than B?

121. A's share of C. P. R. stock sells for $233. The stock pays a yearly dividend of $10. What rate of interest will an investment in the stock yield?

122. On a map one inch represents 4 miles; how many acres would be represented by a rectangle 1·5 inches by 1·25 inches?

123. Find the cost of a herd of cattle sold at 12½% above cost at a profit of $240.

124. Hats cost $43.50 per dozen; 11 sold at $4.50 each, and one for $3. Find the rate of gain on the lot.

125. If I buy oranges at the rate of one cent each, and sell them at the rate of 2 for 5c., what per cent. profit do I make?

126. A dealer sells real estate for a commission of 2%. How much must he sell during the year to secure an income of $75 per month?

127. A real estate agent charges me 2% for selling my property in Toronto. He remits me $5,880. What was the amount of his commission?

128. An agent who sold 150 lots at $233⅓ each, charged $262.50 for his services. What rate of commission did he get?

129. A set of books whose catalogue price is $100 can be bought at a discount of 10% and 5% off for cash. How much less than the catalogue price will they cost?

130. A man had 50 bushels of wheat and sold 20% of it. What per cent. of the portion left is the portion sold?

131. A farmer sold 15% of his wheat to one dealer, and 25% to another, and had 30 bushels left. How much wheat had he originally?

132. A man after spending a month in Muskoka finds his weight to be 210 lb., which is an increase of 5%. What was his weight before he went to Muskoka?

133. A farmer increased his flock of sheep by 12½%, and then had 900. How many had he at first?

134. A paper-hanger estimates the number of rolls of paper required to paper a room by finding the total area of the walls, deducting 20 sq. ft. for each door and window, and dividing the remainder by 33 sq. ft. Find the cost of papering the walls of a room 14 ft. 6 in. long, 13 ft. 4 in. wide, and 12 ft. high, with paper costing 75c. a roll, and a border costing 10c. a yard, if there are 4 windows and 3 doors, and it costs 15c. a roll for hanging.

135. A clerk's salary was $640; he spends 75% of it. Suppose his salary to be increased 30% and his expenses to be increased 40%, find the amount he will then save each year.

136. An agent gains 9c. a lb. by selling twine at 25% above cost. What did it cost him?

137. I owe $1,000, and on February 4th, 1911, give my creditor in payment my note at three months. I wish the face of the note to be such that it will pay the debt if discounted at a bank at 7 per cent. For what amount must I draw the note?

133. Find the selling price of goods bought at $88.65 and sold at 3⅓% below cost.

139. A coal dealer bought 34,160 lb. of coal at $2.50 per long ton (2,240 lb.). He sold 8 loads, each 1 ton, 4 cwt., 60 lb., at $3 a ton, and the rest at the rate of $3.25 a ton. How much did he gain?

140. A farmer sold 150 sheep, which was 75% of his whole flock. How many sheep were there in the flock?

141. A closed rectangular vessel, formed of metal 1 inch thick, whose external dimensions are 12 inches, 10 inches, and 8 inches, weighs 89 pounds. What would be the weight of a solid mass of metal of the same dimensions?

142. A dealer bought a gross of pencils and sold 36 of them. What per cent. of his pencils remain unsold?

143. Change the following fractions to others having 100 for the denominator:

(*a*) $\frac{1}{4}$; $\frac{3}{5}$; $\frac{3}{20}$; $\frac{3}{4}$; $\frac{1}{50}$; $\frac{1}{10}$; $\frac{1}{8}$; $\frac{5}{8}$; $\frac{3}{16}$; $\frac{4}{25}$.

(*b*) Read as hundredths: ·05; ·186; ·33$\frac{1}{3}$; ·27$\frac{1}{4}$; ·2725; ·144.

144. From a flock of 60 sheep 10% were sold. What fraction of the number left is the number sold?

145. A man had $1,500 in the bank and drew out 40% of it. What per cent. of the amount left was the amount drawn out?

146. A farmer bought 11 cows for $253, and after keeping them for 17 weeks at a cost of $1.75 per week, he sold them for $48 each. How much did he gain or lose by the transaction?

147. A bankrupt owes $40,000 to A and $2,500 to B. His assets amount to $650. How much should each debtor receive?

148. A merchant fails and has assets sufficient to pay his creditors 80c. on the dollar. How much will a creditor receive to whom the merchant was indebted for $784?

149. A bankrupt owes A $3,500, B $3,275. His assets amount to $4,403.75; how many cents on each dollar of debt will he be able to pay?

150. A bankrupt owes $6,700 altogether, and has assets worth $3,350. How much should a creditor receive to whom he owed $500?

151. A merchant bought 40 gal. of spirits at $1.50 a gal. He added 10 gal. of water and sold the mixture at $1.50 a gal. Find his gain.

152. Write a negotiable promissory note signed by James Fox for $875.60 for 90 days payable to yourself.

153. A dealer bought 65 lawn mowers at $4.25 each, and sold them at 3.87\frac{1}{2}$ each. What per cent. did he lose?

154. A merchant marked his goods at an advance of 30% on cost, but afterwards sold them at a discount of 30% on the marked price, for $182. Find his loss on the transaction.

155. Divide $4,669 among three men so that the first gets $5 as often as the second gets $7, and the third $11.

156. Assuming a bushel equal to 2218·192 cu. in., estimate the number of bushels of wheat in a bin 12 ft. 9 in. long, 4 ft. wide, and 6 ft. 8 in. high.

157. A man's income is $900; $500 is exempt from taxation; on the rest he pays taxes at the rate of 18 mills on the dollar. Find the amount of his taxes.

158. A merchant bought 8 bushels of wheat at $87\frac{1}{2}$ cents a bushel and sold it at 77 cents a bushel. How much did he lose on every dollar he paid?

159. If 20 per cent. be lost on a ton of straw sold at $19.20, what is the cost of the straw a ton?

160. Bought 1,000 lb. of butter at 18 cents per lb., and sent it to an agent, who sold it at 21c., and charged a commission of 5 per cent. What was my rate of gain?

161. A farmer has a flock of 940 sheep in three fields. In the first are 20 per cent. of the whole flock; in the second, 40 per cent., and the remainder in the third. How many sheep are there in each field?

162. An asphalt company undertook to pave a street five miles long at $55,000 per mile. If the actual cost be $130 per rod, what is the gain per cent.?

163. I bought 1,100 tons of coal at $\$3\frac{1}{3}$ per ton. I sold 40% of it at a gain of 50%; 40% of the remainder at a gain of 35%; and lost 10% on the rest. What was my actual gain?

164. What is the valuation of my property, if my tax, at the rate of 15 mills on the dollar, amounts to $30?

165. The officers of a certain town find that all the town expenses for the year will amount to $46,000. The assessed value of the property in the town is $2,300,000. What is the rate of taxation?

166. Find the interest on:

(*a*) $450 from Aug. 10th to Nov. 8th, 1885, at 6% per annum;

(*b*) $720 from Jan. 25th to April 7th, 1885, at 7% per annum;

(*c*) $960 from Feb. 3rd to March 19th, 1884, at 8% per annum;

(*d*) $540 from April 8th to May 10th, 1870, at 9% per annum;

(*e*) $900 from Feb. 12th to March 4th, 1891, at $7\frac{1}{2}$% per annum.

167. What per cent. of 3 is $\frac{2}{3}$; of $\frac{4}{9}$ is $\frac{2}{9}$?

168. A drover sold 250 sheep for $1,150, which was 15% more than they cost. What was the cost of the sheep a head?

169. A circular ventilating tube, 14 inches in diameter, is delivering a current, with a velocity of 5·32 feet a second, into a hall 52 feet long, 35 feet wide, and $16\frac{1}{2}$ feet high. Find,

to the nearest second, how long it will take for the air of the hall to be renewed in this way.

170. A borrowed $56.50 from B, on May 1st, and agreed to pay interest at the rate of 6% per annum. Find the date on which A paid the loan, if the interest amounted to $2.25.

171. The interest on $109.50, for 110 days, is $2.30. Find the rate.

172. Eighteen per cent. of wheat is lost (as bran, etc.) in grinding it into flour, and the weight of bread is 133⅓ per cent. of the weight of flour used to make it. How many 2-lb. loaves can be made from 10 bushels of wheat?

173. Find the interest on $68.40, from April 20th to Oct. 3rd, at 6½% per annum.

174. A retail merchant bought 4 dozen men's hats at $25 a dozen. He sold all but seven at $3 each. He gave these away free at the end of the season. Find his gain or loss on the transaction.

175. Find the compound interest on $1 for four years at 6% per annum, compounded yearly.

176. An insurance company insured a block of buildings for $200,000 at 75c. per $100, and re-insured $60,000 with another company at 60c. per $100. What amount of premium did it receive, and what amount did it pay out?

177. Beneath a house, 40 feet long and 28 feet wide, is a stone foundation, 1½ feet thick and 5 feet high. How many cubic yards of stone does it contain?

178. Make a table, showing the fractional equivalents of:

$1\frac{1}{4}\%$, $1\frac{2}{3}\%$, $2\frac{1}{2}\%$, $3\frac{1}{3}\%$, $6\frac{1}{4}\%$, $8\frac{1}{3}\%$, $9\frac{1}{11}\%$, $11\frac{1}{9}\%$,
$12\frac{1}{2}\%$, $14\frac{2}{7}\%$, $16\frac{2}{3}\%$, $18\frac{3}{4}\%$, 20%, 25%, $33\frac{1}{3}\%$, $37\frac{1}{2}\%$.

179. A real estate agent sold a farm of 212 acres at $115 an acre, and charged 2% commission. What was the amount of his commission?

180. My Winnipeg agent buys me 4,500 bushels of wheat at 83⅓ cents a bushel. How much should I remit him to cover the cost of the wheat and his commission at 5%?

181. Interest, at the rate of 4% per annum, payable yearly, means that the amount of interest due at the end of each year should be paid to the lender, whether the principal be due then or not. When the amount due as interest is not actually paid at the end of the year (and, in many cases, it is not), it is added to the principal. The whole amount of interest for a number of years reckoned in that way is called compound interest. Find the compound interest on $1,000, for four years, at 6% per annum, payable yearly.

182. A man had $630 on deposit in a bank; he withdrew 30% of the deposit, and with 15% of the amount withdrawn, purchased a suit of clothes. What was the cost of the clothes?

183. A man died, and left an estate valued at $160.000. The Government received 4% of it for succession duties. The remainder was divided among his wife, son, and two daughters. The wife received as much as the children together, and the son as much as the two daughters together. Find the share of each child.

184. A produce dealer paid $160 for apples, $45 for onions, and $60 for potatoes. He sold the apples at a gain of 25%, the onions at cost, and the potatoes at 95% of their cost. Did he gain or lose, and how much?

185. A merchant bought 4 dozen men's hats at $22 a dozen. He sold 3 dozen of the hats at $3 a hat, and one half a dozen at $2.50 each, and the others at $1.50 each. Find the amount he gained by the transaction, and the rate of gain.

186. A man deposits $120 in a savings' bank, the bank pays interest at the rate of 4% per annum, and adds the interest to the principal at the end of each year. How much will the man have to his credit at the end of three years?

187. Find the L.C.M. of:

(*a*) 25 35 56,
(*b*) 48 64 72,
(*c*) 85 51 255,
(*d*) 180 216 120,
(*e*) 100 110 440.

188. Simplify:

(*a*) $(9\frac{7}{10} - 4$[illegible]$) - ($[illegible]$ + 2$[illegible]$)$,
(*b*) 3[illegible]$ + 2\frac{1}{5} + 5$[illegible]$ - 3\frac{1}{8}$,
(*c*) [illegible] of 7[illegible] of $9\frac{9}{10}$ of [illegible].
(*d*) $2\frac{7}{24} +$ [illegible].

189. Two trees have shadows 27 ft. 6 in., and 33 ft. respectively. The shorter tree is 35 ft. high. Find the height of the longer.

190. Find the simple interest on:

(*a*) $325 for 2 yr. at 4% per annum,
(*b*) $275.50 for 4 yr. at $3\frac{1}{2}$% per annum,
(*c*) $845.75 for $4\frac{1}{2}$ yr. at 5% per annum.

191. Add:

(*a*) $\frac{7}{10} + \frac{11}{12} +$ [illegible] $+$ [illegible],
(*b*) [illegible] $+$ [illegible] $+$ [illegible] $+$ [illegible],
(*c*) $\frac{5}{11} + \frac{2}{15} +$ [illegible] $+ \frac{17}{60}$,
(*d*) [illegible] $+ \frac{5}{72} +$ [illegible] $+ \frac{7}{24}$.

192. Find the square root of 107·101801.

193. A room is 20 ft. long, 10 ft. wide, and 10½ ft. high. Find the cost of painting the walls at the rate of 4d. a square foot, the area of the windows and door being taken as 25 sq. ft.

194. Multiply £14 7s. 10½d. by 874.

195. Find to the nearest ounce the weight of water that can be contained in a rectangular basin 2 ft. 6 in. long, 1 ft. 4½ in wide and 7⅞ in. deep, the weight of a cu. ft. of water being 1,000 oz.

196. The population of England and Wales has increased every 10 years from the year 1860 by the following percentages: 18·21, 14·36, 11·65, 12·17, each percentage being reckoned on the population at the commencement of the 10-year period concerned. If the population in 1900 was 32,527,843, what was it in 1860? In 1880?

197. Simplify: $\dfrac{2{\cdot}0857-(1{\cdot}069+1{\cdot}0042)}{3{\cdot}314+({\cdot}5 \text{ of } 18{\cdot}372)}$ of ·8875.

198. At an auction sale the terms are: All bids under $10 are to be net cash, but on all bids of $10 and over there will be allowed a discount of 8% for cash. Which is the better bid, $9.75 or $10.25?

199. A High School graduate is said to be able to produce 1½ times as much wealth as a graduate of the Public School.

(*a*) If the annual earnings of the latter are $550 a year, what will be the difference in the earnings of the two at the end of 30 years?

(*b*) If the former attends school 4 years longer than the latter at an average annual expense of $125 a year, how much will the former earn more than the latter in 40 years from the time he enters High School, taking for granted that the latter was at work during the whole period?

(*c*) What then was the money value of each day's attendance at the High School, taking the school year to be 200 days?

200. A boy untrained in the use of tools is said to waste 1¼ in. in the length of a board for each cut he makes with the saw. What is the value of the loss in making 150 cuts in 1¼ in. lumber 6 in. wide which sells for $30 a thousand?

201. If a kitchen cupboard saves a woman one hundred and twenty 22-in. steps daily, how many miles will it save her in 10 years?

202. A machine that cost an agent $135 was left out in the sun and rain for 15 months and was then sold for $75. If money is worth 6% per annum find the total loss due to the agent's carelessness.

203. A flock of 50 hens are fed daily $3\frac{1}{2}$ lb. of wheat worth 90c. a bushel. If the hens average 90 eggs a year each, and eggs sell at an average of 18c. a dozen, what is the year's profit?

204. To destroy insects and rot a farmer sprayed a two-acre field of potatoes three times with a Bordeaux mixture containing Paris-green. The expense was as follows: 48 lb. lime at $1\frac{1}{2}$c. a pound; 48 lb. copper sulphate at 25c. a pound; 3 lb. Paris-green at 28c. a pound; and $8.50 for labour. The yield was 160 bu. to the acre and the potatoes sold at 55c. a bushel. If an unsprayed acre yields [illegible] bu. what is the value an acre for spraying?

205. If a cubic foot of oak weighs 64 lb. what is the weight upon a wagon loaded with 24 oak ties $8\frac{1}{2}$ ft. long, 9 in. wide, and $7\frac{1}{2}$ in. thick?

206. If the water running from a piece of land contains $\frac{1}{4}$ lb. sediment for every 100 gal. of water, how much soil will be carried away from a 40-ac. field after a 2-in. rainfall, if $\frac{3}{4}$ of the water runs off and a gallon contains $277\frac{1}{4}$ cubic inches?

207. A town has a population of 250,000, and a water supply of 30 gallons per head per day is required. A reservoir is provided to hold a supply of water from Dec. 31st to June 30th. What is its surface area if it has an average depth of 20 feet and if a cubic foot will contain $6\frac{1}{4}$ gallons of water?

208. How much greater is $\frac{3}{7}$ of £73 3s. than $\frac{5}{8}$ of £72 15s.?

209. A man's step equals 30 inches; he takes 120 steps to the minute. How far, in miles, will he go in 6 hr. and 45 min.?

210. Make out a bill for 14 lb. steak at 1s. 1d. a lb.; [illegible] lb. chops at 10d. a lb.; $5\frac{1}{2}$ lb. lamb at 1s. 2d. a lb.; and 7 lb. suet at $4\frac{1}{2}$d. a lb.

211. A green-grocer buys 17 pk. 1 gal. of currants for £2 6s. 8d. and sells them at 6d. a quart; find his total gain.

212. How many gas-pipes each 9 ft. long will it require for a main extending 6 miles?

213. How much money must I lend for 4 yr. 7 mo. at simple interest so that I may receive $77 interest?

214. Simplify: $\frac{3\frac{1}{3}+3\frac{3}{4}}{9-\frac{1}{4}}+\frac{\frac{3}{8}+13\frac{1}{8}}{18\frac{1}{4}-6\frac{1}{2}}+\frac{27}{105}$.

215. Simplify: $\frac{\cdot01256+2\cdot75+\cdot004}{\cdot0005}+466\cdot88$.

216. Find the cost of carpeting a room $19\frac{1}{2}$ ft. long and $17\frac{1}{2}$ ft. wide with carpet 28 in. wide costing $1.25 a yard.

217. A good ration for a dairy cow is 10 bu. of corn and 10 bu. of oats ground together, and 1 ton clover. If this

amount should suffice for 75 days, find at current prices the cost of feeding a cow for 225 days, and pasturing for the rest of the year at $1.50 a month.

(*a*) A cow gives each day $3\frac{1}{2}$ gallons of milk weighing $8\frac{3}{5}$ lb. a gallon. If the milk tests 3.8 per cent. butter fat, how much butter fat does she yield a week?

(*b*) Which would be the more profitable to sell, the milk at 5c. a quart or the butter fat at 25c. a pound?

218. A flock of 100 hens averages 90 eggs each a year. If the average price of eggs is 20c. a dozen and if it takes 12 bu. of corn at 45c., 5 bu. of oats at 35c., and $7 worth of other feed to keep the flock for one year, what is the year's profit?

219. Make out a market ticket for the following 5 loads of oats at 40c. a bu.:

Gross weights—2900 lb., 2842 lb., 2736 lb., 2937 lb., 2854 lb.
Wagon weights—1187 lb., 1264 lb., 1235 lb., 1010 lb., 1224 lb.

220. If 2 bu. of ear corn will fill 5 cu. ft. how many bushels of ear corn will it take to fill a crib 10 ft. wide, 32 ft. long, and averaging 10 ft. high?

221. If 5 bu. of small grain will fill 4 cu. ft. how many bushels of oats are there in a bin $5\frac{1}{2}$ ft. wide, 8 ft. high, and 22 ft. long?

222. If 3 bu. of apples will fill 4 cu. ft. how many bushels will it take to load a freight car 8 ft. wide, 32 ft. long, and 4 ft. deep?

223. Find the amount of the following thrashing bill:

2600 bu. of oats at 2c. a bu.,
1620 bu. of wheat at 4c. a bu.,
2800 lb. of coal at $5.40 a ton,
4 teams for $2\frac{1}{2}$ days at $3.80 a day,
8 men for $2\frac{1}{2}$ days at $1.80 a day.

224. (*a*) What will it cost per acre to grow wheat allowing for ploughing 3 acres a day at $3.75 a day, for harrowing (twice) 15 acres a day at $3.75 a day, for seeding 15 acres a day at $3.75 a day, 6 pk. of seed per acre at $1.10 a bu., 75c. an acre for cutting, 4 acres a day for shocking at $1.80 a day, and 5c. a bu. for threshing on a yield of 20 bu. to the acre?

(*b*) What would be the net return on 30 acres of the above wheat, if the wheat sold at 90c. a bu. and the land was rented at $5 an acre?

225. Find the total cost of ploughing 7·025 ac. at 12·5s. an acre, mowing 3·75 ac. at 1·125s. an acre, and drilling 5·5 ac. corn at 1·5s. an acre.

226. Express $\frac{1}{3}+\frac{1}{6}+\frac{1}{4}+\frac{3}{56}$ as a decimal.

227. Reduce 231 yd. to the decimal of $\frac{1}{4}$ of a mile.

228. Find the cost of painting both sides of a high board fence 150 ft. long, and 8 ft. high at 6½c. a sq. yd.

229. Find the number of bricks 9 in. by 4½ in. needed for a walk 4½ ft. wide and 5 yd. long.

230. A cow needs 600 cu. ft. of air. How many cows may be placed in a stable 75 ft. long, 12 ft. wide, and 14 ft. high?

231. Simplify: $\frac{1\frac{4}{17} \times 6\frac{4}{5}}{3\frac{3}{5} - 1\frac{9}{25}} + \frac{3}{11}$ of $(2\frac{3}{5} - \frac{3}{4}) - \frac{1\frac{3}{5}}{12} + (\frac{3}{5}$ of $\frac{6}{11})$.

232. Dress goods are offered at 50c. a yd. in 36-in. widths or 80c. a yd. in 54-in. widths. If it takes 6½ yd. of 36-in. goods to make a dress, what would it cost to make the dress from 54-in. goods?

233. How many yards of sheeting 2¼ yd. wide must be bought to make 9 sheets, if each sheet when finished is to be 2½ yd. long with a 1½ in. hem at one end and a 2 in. hem at the other end and 2½ in. to be allowed for shrinkage?

234. The average man loses 6 days a year from minor ailments, 90% of which could be prevented by care. If, in Canada, there are 3,500,000 workers earning on an average $1.25 a day, estimate the total annual loss due to carelessness.

235. From good seed the yield is 33⅓% more than from poor. Good seed costs $1.04 a bushel, while the poor is bought for 80c. For each acre 1¾ bu. is required. Find the profit in sowing good seed on 12 acres of land the yield being 24 bu. to the acre.

236. A grocer pays $1.20 a bu. for apples, which he retails at 20c. a dozen. The apples average 90 to the bushel. What per cent. does he make on the sale of each bushel if 10% of the sales is the cost of handling?

237. The average work-horse requires daily about 2 lb. of protein. Oats contains 9.2% of protein and corn 7.8%. If oats are worth 35c. a bushel and corn 45c. which is the cheaper food so far as protein alone is concerned?

238. If 1 bu. of grain produces 11 lb. of flesh, what will it cost to increase the weight of a pig from 100 lb. to 254 lb., grain being worth 75c. a hundredweight?

239. The annual average loss to a farmer because of bad roads is estimated at 75c. an acre. What is the loss to a farmer who owns 125 acres?

240. How many bunches of shingles 4 in. wide and laid 4 in. to the weather will it take to cover your school-house, the row at the eaves being double.

241. Hanover, Jan. 8, 1915.

$255.00

Four months after date for value received I promise to pay James Grange or order Two Hundred and Fifty-five Dollars with interest at 6%.

Thos. L. Barton

Find the proceeds of this note when discounted at the bank on Feb. 12th, 1915, at 7%.

242. (*a*) Wheat removes from the soil 2·4% of its weight in nitrogen, ·9% phosphoric acid, and ·6% potash. What will be the total amount of each and all of these three elements removed from a field of 25 acres which produces 26 bu. to the acre?

(*b*) How much nitrate of soda which contains 16% nitrogen would it require to replace the nitrogen drawn from the soil of the above field?

243. A farmer has two cows, one of which gives 8,000 lb. of milk yearly, and the other 5,000 lb. The milk of the first contains 3·2% of butter fat, that of the other 5·6%. Which is the more valuable cow, butter fat being worth 25c. a pound?

244. How many pounds of cream containing 25% of butter fat are there in 400 lb. of milk containing 4% of butter fat?

245. At 20c. a pint (1 lb.) what is the value of cream containing 30% butter fat taken from 250 lb. of milk containing 4·2% butter fat?

246. A fruit grower bought 700 pounds of arsenate of lead at $7.50 a hundredweight to spray for apple-worms. He used only 60% of it and the balance was left over till the next year. If money is worth 6% per annum how much would he gain or lose by buying in each of the two years exactly what he needed at 8c. a pound?

247. If a ton of hay requires a space of 15 cu. yd., how many tons will a mow 45 ft. long, 30 ft. wide, and 21 ft. high hold?

248. A liveryman started business with the following equipment: 3 hacks valued at $425 each, 8 horses valued at $225 each, one carriage valued at $180, 3 sets heavy harness worth $65 a set, and 2 sets single harness worth $30 each, robes, etc., worth $48, drivers' outfit worth $56. He pays rent at $45 a month, $2 a day for each of 4 drivers and 50c. a day to feed each horse, and the annual premium of $\frac{5}{8}$% on $3,000 insurance. If the average daily earnings of each horse is $2.50 how long will it take the business to pay for the equipment?

249. An automobile agent agreed to sell two cars a month. The cars were valued at $540 each and the agent was to receive $100 for each car sold. He paid $20 freight on each car sold, $25 for an annual license for himself, $30 for advertising, $65 for gasoline and repairs for a demonstration car which he purchased for $440 and sold at the end of the year for $220. Find the agent's annual profits.

250. A grocer makes the following profit on his goods: Tea and coffee, 40%, fruits and vegetables, 33⅓%, butter and eggs, 20%, sugar, 50c. a hundredweight, and other articles 25%. His average daily sales are 75 lb. coffee at 30c., 50 lb. tea at 45c., 100 lb. butter at 25c., 70 doz. eggs at 18c., fruit and vegetables $75, sugar 375 lb., and other articles $50. Find his average weekly profits.

251. To make a 20 oz-loaf a baker uses 2 tablespoonfuls of lard at 16c. a lb., ¼ pt. milk at 8c. a quart, 3 cups flour at $6.00 a bbl. and ⅓ cake of yeast at 2c. a cake. A tablespoonful of lard weighs ½ oz. and a cup of flour ½ lb. The bread sells at 10c. a loaf. How much profit is there in making 6 bbl. of flour into bread if 8c. worth of salt is added to each bbl. and the total cost of fuel and other incidental expenses is $2.50?

252. A street railway company has a line on which 30 cars make a round trip every hour from 6.00 a.m. to 12.00 p.m. The average number of people riding each way is 30, of whom 20% ride on transfers. The fare is 5c. each way. If each conductor and motorman receives 28c. an hour, and the other expenses average 5c. a car an hour, find the company's daily net receipts.

253. A man bought a double house and lot for $4,500. He insures the property for 3 years at 1½% on a policy for $3,600. The taxes were 18 mills on an assessment of $4,000. He sold the property at the end of three years for 16⅔% more than he paid for it and allowed an agent 5% for making the sale. During the time he owned the house he spent $125 in making repairs. Each house rented at $18 a month, but one of them was vacant 5 months. Find the owner's net profit.

254. The rent paid for a shoe-shining establishment is $1,500 a year. The manager gets $25 a week and each of three helpers $15 a week. The other expenses are $15 a month for light and fuel and $1.50 a week for supplies. The price for shining black shoes is 5c. and 10c. for tan. The average daily work for a year of 305 days is 325 shines, 40% of which is for tan shoes. Find the year's profit.

255. Make out a bill for the following furnished by a plumber: 19 ft. excavating at 15c. a linear foot, 75 ft. soil pipe at 35c., 10 ft. iron pipe at 16c., trap $25, 165 lb. lead at 4c., 18 lb. oakum at 12½c., 25 lb. solder at 29c., 9 gal. gasolene at 22c., 56 ft. half-inch pipe at 6¾c., fittings $17, 3 tanks at $14 each, 2 sinks at $11.25 each, 4 sets trays at $10.50 a set, 158 hours' labour at 60c. an hour. Discount for cash in 10 days 5%.

256. A contractor agreed to build four miles of roadway for $9,500 a mile. The road was to be 16 ft. wide and built of

crushed stone 9 in. thick. There were 2,400 cu. yd. of dirt to be moved at 35c. a cu. yd. and the contractor required 75 tons of coal worth $4.25 a ton, 8,400 ft. of lumber at $36 a thousand, 2,200 sacks of cement at $1.65 a bbl., each barrel containing 4 sacks, and other material valued at $450. If it cost the contractor $3.50 a cu. yd. to furnish and lay the crushed stone, what were his total profits?

257. A shoe dealer sells shoes at $3.50, $4.00, and $5.00, respectively, and makes a profit of 25%. His average daily sales are 30 pairs consisting of an equal number of each grade. His expenses are $25 a week for a manager, $15 a week for one clerk, $1.25 a week for extra help, $125 a month for rent, and $18 a month for other expenses. Find his annual profit if his store is open for 305 days.

258. A farmer has a field ¼ of a mile long and ⅛ of a mile wide in which he sows wheat. He hires a plough which can turn 10 acres a day at $25 a day, and a drill which can sow 8 acres a day at $5 a day. He sows 6 pecks of wheat to the acre and pays 90c. a bu. for it. Harvesting costs $2.25 an acre, thrashing 4c. a bu. and other incidentals 10c. a bu. The yield was 22 bu. an acre and the grain was sold at 95c. a bu. Find the farmer's *net* receipts from the field.

259. A man's farm is assessed for $3,600, his personal property for $800, and his income is $1,500, of which $800 is exempt. What tax does he pay at 15 mills on the dollar?

260. At 10c. a square foot what will it cost to cement the floor and walls of a cellar 16 ft. long, 14 ft. wide, and 8 ft. deep?

261. Make out a bill for the following lumber at $30 a thousand:

16 pieces 2 in. × 2 in × 10 ft.,
15 pieces 4 in. × 5 in × 12 ft.,
28 pieces 2½ in. × 8 in. × 14 ft.,
100 pieces 1½ in. × 6 in. × 10 ft.,
60 pieces 1 in. × 8 in. × 15 ft.

262. A farmer drains a field 60 rods long by 28 rods wide with tile 12 in. in length and costing $4.50 a hundred. He lays 4 lines of the tile the full length and 3 the full width of the field. He pays 7½c. a yard for digging the ditches and laying the tile. If, before draining, the field would produce 33 bu. of oats an acre, but afterwards produces 50 bu., in how many years would it take the increase of the crop to pay for the tiling, oats being worth 42c. a bu.?

263. Find the interest of the current year on:

(*a*) $60 at 4½% from May 3rd to Sept. 10th.
(*b*) $125 at 5% from June 10th to Oct. 8th,
(*c*) $259.50 at 5½% from Apr. 3rd to Aug. 20th.

264. The most important plant foods found in the soil are nitrogen, phosphoric acid, and potash. The first is worth 20c., the second 5c., and the third 5c. a pound. To enrich his soil, a farmer ploughs under clover hay and another oat straw. Which is the better fertilizer if a ton of clover hay contains 41 lb. nitrogen, 7 lb. phosphoric acid, and 44 lb. potash, while oat straw contains 12 lb. nitrogen, 4 lb. phosphoric acid, and 24 lb. potash? How much per cent. is the one better than the other?

265. A dry-goods dealer bought $2,460.30 worth of goods which he marked at 33⅓% above cost. After selling half of it, he reduced the marked price 10%. How much did he gain? How much per cent.?